Math 113 Exam 2 Practice

October 5, 2009

Exam 2 will cover sections 7.1-7.8, and Appendix G. Note that this exam covers 1 and 1/2 sections that were also covered in exam 1. This is not a typo. For completeness, 7.1 and 7.2 are again included on this sheet. Note that substitution, while not specifically included in chapter 7, may still be needed.

This sheet has three sections. The first section will remind you about techniques and formulas that you should know. The second gives a number of practice questions for you to work on. The third section give the answers of the questions in section 2.

Review

7.1: Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

- Integration by parts is most often useful when integrating a function of the form $x^n e^x$, $x^n \sin x$, $x^n \cos x$, $x^n \ln x$. If possible, you want to choose u to be a function that becomes simpler when differentiated, and dv to be a function that can be readily integrated. This usually means you should choose $u = x^n$. (But in the case $x^n \ln x$, choose $u = \ln x$).
- Integration by parts is also useful for integrating inverse functions such as $\sin^{-1} x$, $\tan^{-1} x$, $\ln x$ or functions involving these as factors. In this case, you should choose $u = \sin^{-1} x$, $u = \tan^{-1} x$, or $u = \ln x$ accordingly, even if there are no other factors in the integrand (i.e., you can set $dv = dx$).

7.2: Trigonometric Integrals

- For $\int \sin^m x \cos^n x\,dx$:
  If n is odd, save one $\cos x$ and convert the rest to sin using $\cos^2 x = 1 - \sin^2 x$
  If m is odd, save one $\sin x$ and convert the rest to cos using $\sin^2 x = 1 - \cos^2 x$
  If both m and n are even, use the identities $\sin^2 x = \frac{1-\cos 2x}{2}$ and $\cos^2 x = \frac{1+\cos 2x}{2}$.
- For $\int \tan^m x \sec^n x\,dx$:
  If n is even, save one $\sec^2 x$ and convert the rest to tan using $\sec^2 x = \tan^2 x + 1$
  If m is odd, save a $\sec x \tan x$, and convert the rest to sec using $\tan^2 x = \sec^2 x - 1$.
  If m is even and n is odd, convert everything to sec and integrate by parts with $dv = \sec^2 x$. (This last case will require "solving" for the desired integral.)
- For $\int \tan^n x\,dx$, convert one $\tan^2 x$ to $\sec^2 x - 1$ and split the problem into two integrals.
- $\int \sec x\,dx = \ln|\sec x + \tan x| + C$.
- A similar strategy applies for $\int \cot^m x \csc^n x\,dx$.

7.3: Trigonometric Substitution

- If the integrand involves $\sqrt{a^2 - b^2x^2}$, use $x = \frac{a}{b}\sin\theta$.
- If the integrand involves $\sqrt{b^2x^2 + a^2}$, use $x = \frac{a}{b}\tan\theta$.
- If the integrand involves $\sqrt{b^2x^2 - a^2}$, use $x = \frac{a}{b}\sec\theta$.

- If the integrand involves $\sqrt{ax^2+bx+c}$, complete the square to get it into the form $\sqrt{a(x-h)^2+k}$. After factoring out the a and applying the substitution $u = x - h$, the integrand will then fit one of the three forms above.

- Avoid using a trigonometric substitution when a regular u-substitution is possible.

7.4: Integration of Rational Functions by Partial Fractions

Rational functions consist of fractions of polynomials. We can split rational functions into simpler pieces by partial fractions. Remember that partial fraction decompositions are based on linear and quadratic factors in the denominator. For each linear factor, we have a term with a constant in the numerator and the factor in the denominator. For each irreducible quadratic, we have a term with a linear function in the numerator and the quadratic in the denominator. For example,

$$\frac{x^2-x+2}{(x-1)(x+2)(x^2+x+3)} = \frac{A}{x-1} + \frac{B}{(x+2)} + \frac{Cx+D}{x^2+x+3}. \tag{1}$$

We just need to determine the values of A, B, C and D. This is done by plugging in values for x: You need to plug in as many numbers as you have constants. Using some numbers, (like -2 and 1 in this case) makes your life easier, but any four numbers will do. Notice that if we multiply equation 1 by the denominator on the left side, we get

$$x^2-x+2 = A(x+2)(x^2+x+3) + B(x-1)(x^2+x+3) + (Cx+D)(x-1)(x+2). \tag{2}$$

Letting $x = -2$ in equation 2 gives $8 = -15B$, so $B = -8/15$. Letting $x = 1$ gives $2 = 15A$, so $A = 2/15$. Letting $x = 0$ gives $2 = 5A - 3B - 2D$. Knowing A and B helps us to find D. Finally, if $x = -1$, $4 = 9A - 6B + 2C - 2D$, allowing us to solve for C.

Remember that repeated factors must give repeated terms with increasing exponent in the denominator. For example,

$$\frac{x^3+2x^2+2x-5}{(x-2)^3(x^2+9)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3} + \frac{Dx+E}{x^2+4} + \frac{Fx+G}{(x^2+4)^2}$$

Finally, remember partial fractions only works if the degree in the numerator is less than the degree in the denominator. Otherwise, you need to divide and use partial fractions on the remainder.

7.5 Strategy for Integration

You may have noticed that we have really only used two techniques of integration in chapter 7: substitution and integration by parts. Everything else is just manipulation of those two techniques. Luckily, most integrals that require a specific technique have patterns that help remind us. Those patterns have been discussed earlier. You are expected to know them.

A general strategy for attacking unknown integrals is as follows:

1. Can substitution be used to simplify the integral? If so, do this first. (Note that in this section, substitution is often just the first step. After the substitution, some other technique needs to be applied.)

2. If you cannot use substitution, or you have used it but the integral is still not simple enough, look for patterns.
   (a) If the integral is a product of trig functions, use the patterns we learned in 7.2 to tackle it.
   (b) If the integral has a sum or difference of squares in the integrand, use trig substitution.
   (c) If the integrand is a rational function, use partial fractions.

3. If you cannot see one of the above patterns, and cannot use substitution, try integration by parts.

7.7 Approximate Integration

In this section we concentrate on two things: How can we efficiently approximate the solution to a definite integral, and how can we approximate the error.

You are expected to know how to calculate the following approximations of a definite integral $\int_a^b f(x)\,dx$:

(a) the left and right hand sum

Left Hand Sum: $L_f = \Delta x \sum_{k=1}^{n} f(x_{k-1})$

Right Hand Sum: $R_f = \Delta x \sum_{k=1}^{n} f(x_k)$

(b) the Trapezoid rule

$$T_f = \frac{\Delta x}{2}\left(f(a) + f(b) + 2\sum_{k=1}^{n-1} f(x_k)\right)$$

(c) the Midpoint rule

$$M_f = \Delta x \sum_{k=1}^{n} f\left(\frac{x_k + x_{k-1}}{2}\right)$$

(d) Simpson's rule

$$S_f = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{n-1}) + f(x_n)\right].$$

Remember, n must be even to use Simpson's rule.

It is unlikely that you will need to calculate the left or right hand sums, but you will still need to know about them.

You are expected to know how each of these rules behave based on standard properties of the function (monotonicity, concavity, etc). For example, it is well known that the left hand sum is a lower bound and the right hand sum is an upper bound of the definite integral of increasing functions. What behavior is true for Midpoint and trapezoid?

You are expected to know the error estimates of Trapezoid, Midpoint and Simpsons:

Trapezoid

$$|E_T| \le \frac{K(b-a)^3}{12n^2}$$

with $|f''(x)| \le K$ on $[a, b]$.

Midpoint

$$|E_M| \le \frac{K(b-a)^3}{24n^2}$$

with $|f''(x)| \le K$ on $[a, b]$.

Simpson's

$$|E_S| \le \frac{K(b-a)^5}{180n^4}$$

with $|f^{(4)}(x)| \le K$ on $[a, b]$.

You can be expected to use the error estimates in one of two ways: to estimate the error of the calculation for a particular value of n, or to find a value for n that gives an error no more than some stated value.

7.8 Improper Integrals

Remember, there are two types of improper integral:

- Infinite length: Integrals of the type

$$\int_{-\infty}^{a} f(x)\,dx, \quad \int_{a}^{\infty} f(x)\,dx.$$

- Unbounded integrand. Integrals of the type

$$\int_{a}^{b} f(x)\,dx, \quad \int_{c}^{a} f(x)\,dx$$

where f has an infinite discontinuity at a.

Remember, in each type, there is a "problem" that a definite integral cannot handle. We remove the problem by turning it into a limit:

$$\int_{a}^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_{a}^{b} f(x)\,dx$$

$$\int_{c}^{a} f(x)\,dx = \lim_{b\to a^-}\int_{c}^{b} f(x)\,dx.$$

Some things to remember when calculating improper integrals:

- Do not forget to set up an improper integral as a limit. You will likely have points deducted if you do not. It is the only way for the grader to tell that you know what you are doing.
- Watch out for infinite discontinuities in the middle of the interval. You must split the integral at the discontinuity in that case.

Appendix G

This section is somewhat of a departure from Chapter 8. This section is basically included to fill some gaps in your mathematical education. We can understand the natural logarithm better if we define it as an area integral, than if we just define it as the inverse of the natural exponential. In some ways, it is a more "natural" way to develop these two functions.

As you prepare for the exam, you should concentrate on two aspects:

- Use the area integral form of the natural logarithm to approximate it (by approximating the area).
- Be able to prove some of the properties of the natural logarithm using the area integral form of the natural logarithm.

Questions

Try to study the review notes and memorize any relevant equations **before** trying to work these equations. If you cannot solve a problem without the book or notes, you will not be able to solve that problem on the exam.

For problems 1 to 30, evaluate the integral.

1. $\int x \cos x \, dx$
2. $\int_0^{\frac{\pi}{2}} x \sin x \, dx$
3. $\int_0^1 x^2 e^x \, dx$
4. $\int_0^1 \sin^{-1} x \, dx$
5. $\int 2x \tan^{-1} x \, dx$
6. $\int \frac{\ln x}{x^2} \, dx$
7. $\int e^x \cos x$
8. $\int_{\sqrt{\frac{\pi}{2}}}^{\sqrt{\pi}} x^3 \sin(x^2) \, dx$ [Hint: First use a u-substitution]
9. $\int_0^{\frac{\pi}{6}} \sin^2 x \cos^3 x \, dx$
10. $\int_0^{\frac{\pi}{3}} \cos^4 x \, dx$
11. $\int_0^{\frac{\pi}{4}} \tan x \sec^2 x \, dx$
12. $\int \tan^2 x \sec^2 x \, dx$
13. $\int \tan^6 x \, dx$
14. $\int \tan^2 x \sec x \, dx$
15. $\int \frac{1}{x\sqrt{x^2+1}} \, dx$
16. $\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}} \, dx$
17. $\int \frac{1}{x^2\sqrt{25+x^2}} \, dx$
18. $\int \frac{x}{\sqrt{x^2+2x}} \, dx$
19. $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x}{\sqrt{x^2+x+\frac{5}{4}}} \, dx$
20. $\int \frac{x}{\sqrt{4x-x^2}} \, dx$
21. $\int_0^1 \frac{2x-1}{x^2-x-2} \, dx$
22. $\int \frac{x}{\sqrt{x^2+2x}} \, dx$
23. $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x}{\sqrt{x^2+x+\frac{5}{4}}} \, dx$
24. $\int \frac{x}{\sqrt{4x-x^2}} \, dx$
25. $\int \frac{6}{x^3+2x^2+x} \, dx$
26. $\int \frac{x^2+x-5}{x^2-1} \, dx$
27. $\int \frac{x-1}{x^3+x} \, dx$
28. $\int \frac{x^3}{x^4+2x^2+1} \, dx$
29. $\int_9^{16} \frac{\sqrt{x}}{x-4} \, dx$ [Hint: Use a rationalizing substitution]
30. $\int_0^{\frac{\pi}{2}} \frac{1}{2-\cos x} \, dx$ [Hint: Use the substitution $t = \tan(\frac{x}{2})$]
31. Estimate $\int_0^4 e^{-x^2} \, dx$ with midpoint, trapezoid, and simpson for n=4 and 8.
32. Calculate a bound on the error for the trapezoid calculation in the last question.
33. How large does n need to be in order for midpoint to have an error no larger than 10^{-5}?
34. How large does n need to be in order for Simpson's to have an error no larger than 10^{-5}?

For questions 35 to 41, evaluate the integral, or show that it diverges.

35. $\int_2^{\infty} \frac{1}{x \ln x} \, dx$
36. $\int_1^{\infty} \frac{\ln x}{x^4} \, dx$
37. $\int_1^{\infty} \frac{1}{(2x+1)^3} \, dx$
38. $\int_0^1 \ln x \, dx$
39. $\int_{\pi/2}^{\pi} \sec x \, dx$
40. $\int_1^{\infty} \ln x \, dx$
41. $\int_1^4 \frac{dx}{x^2-4}$

For questions 42 to 43, use the Comparison Theorem to show whether the improper integral converges or diverges.

42. $\int_1^{\infty} \frac{\tan^{-1} x}{x^2} \, dx$
43. $\int_1^{\infty} \frac{2+e^{-x}}{\sqrt[3]{x^2}} \, dx$
44. By comparing the areas, show that $2(\frac{1}{3}+\frac{1}{5}+...+\frac{1}{2n+1}) < \ln(2n+1) < 2(1+\frac{1}{3}+...+\frac{1}{2n-1})$.
45. Find the equation of the tangent line to the curve $y = 1/t$ that is parallel to the secant line AD, where $A = (1,1)$ and $D = (3, 1/3)$.

46. What is $\log_{\sqrt{10}} 10 + \ln(\frac{1}{e}) + 2\log_5 \frac{5}{2} + \log_{25} 16$?

(a) 5

(b) $\frac{7}{2}$

(c) 3

(d) $\frac{3}{2}$

(e) $3 - \log_5 2$

(f) $5 - \log_5 2$

47. Which one is NOT right?

(a) $\ln x$ means the area under the curve $y = 1/t$ from $t = 1$ to $t = x$;

(b) $\lim_{x\to-\infty} e^x = \lim_{x\to-\infty} xe^x = 0$;

(c) $\lim_{x\to 0+} x \ln x = \lim_{x\to 0+} \sqrt{x} \ln x = 0$;

(d) $\int_1^\infty \frac{1}{x(\ln x)} dx$ is convergent;

(e) $\frac{d}{dx} a^x = (\ln a) a^x$.

Answers

1. $x \sin x + \cos x + C$
2. 1
3. $e - 2$
4. $\frac{\pi}{2} - 1$
5. $x^2 \tan^{-1} x - x + \tan^{-1} x + C$
6. $-\frac{1}{x} \ln x - \frac{1}{x}$
7. . $\frac{1}{2} e^x (\sin x + \cos x) + C$
8. $\frac{1}{2}(\pi - 1)$
9. $\frac{17}{480}$
10. $\frac{\pi}{8} + \frac{7}{64}\sqrt{3}$
11. $\frac{1}{2}$
12. $\frac{\tan^3 x}{3} + C$
13. $\frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} + \tan x - x + C$
14. $\frac{1}{2}(\sec x \tan x - \ln|\sec x + \tan x|) + C$
15. $\ln|x| - \ln|\sqrt{x^2+1} + 1| + C$
16. $\frac{40}{3}$
17. $\frac{-\sqrt{x^2+25}}{25x} + C$
18. $\sqrt{x^2+2x} - \ln|x + 1 + \sqrt{x^2+2x}| + C$
19. $\sqrt{2} - 1 - \frac{1}{2}\ln(\sqrt{2}+1)$
20. $2\sin^{-1}(\frac{x}{2} - 1) - \sqrt{4x - x^2} + C$
21. 0
22. $\sqrt{x^2+2x} - \ln|x + 1 + \sqrt{x^2+2x}| + C$
23. $\sqrt{2} - 1 - \frac{1}{2}\ln(\sqrt{2}+1)$
24. $2\sin^{-1}(\frac{x}{2} - 1) - \sqrt{4x - x^2} + C$
25. $\ln|x| - \ln|x+1| + \frac{1}{x+1} + C$
26. $x - \frac{3}{2}\ln|x-1| + \frac{5}{2}\ln|x+1| + C$
27. $-\ln|x| + \frac{1}{2}\ln|x^2+1| + \tan^{-1} x + C$
28. $\frac{1}{2}(\ln|x^2+1| - \tan^{-1} x + \frac{x+1}{x^2+1})$
29. $2(\ln 5 - \ln 3 + 1)$
30. $\frac{2\sqrt{3}}{9}\pi$
31. $n = 4$**: Midpoint** .8861352469
    Trapezoid .8863185462
    Simpson .8362142646
    $n = 8$**: Midpoint** .8862269183
    Trapezoid .8862268966
    Simpson .8861963468
32. $|f''(x)| < 2$ on $[0, 4]$. $|E| < \frac{32}{3}$.
33. 731
34. $|f^{(4)}(x)| < 12$. $n = 52$.
35. Diverges.
36. $\frac{1}{9}$
37. $\frac{1}{36}$
38. -1
39. Diverges.
40. ∞
41. Diverges.
42. Converges. Make the comparison $\dfrac{\tan^{-1} x}{x^2} < \dfrac{\pi/2}{x^2}$.
43. Diverges. Make the comparison $\dfrac{2 + e^{-x}}{\sqrt[3]{x^2}} > \dfrac{2}{\sqrt[3]{x^2}}$.
44.
45. $y = -\frac{1}{3}x + \frac{2}{\sqrt{3}}$
46. c)
47. d)