Math 113 Exam 4 Practice

Exam 4 will cover 11.1-11.11. Note that even though 11.1-11.4 was tested in exam 3, questions from these sections will also be on this exam. **Refer to Exam 3 practice for the material in those sections**. This sheet covers 11.5-11.11.

This sheet has three sections. The first section will remind you about techniques and formulas that you should know. The second gives a number of practice questions for you to work on. The third section give the answers of the questions in section 2.

Review

Tests for Convergence

We learned about the following tests for convergence:

Divergence Test If $a_n \not\to 0$ then $\sum a_n$ diverges. This is an excellent test to start with because the limit is often easy to calculate. Keep in mind, however, if the limit **is** 0, then the Divergence test tells you nothing. You must try some other test.

p series If you recognize a series as a p series,

$$\sum \frac{1}{n^p}$$

then you can use the fact that a p series converges when (and only when) $p > 1$.

Geometric series We discussed this in the last subsection.

Comparison Test To use the comparison test, we need to have a large group of test series available. We also need to know if these test series converge or not. The most common test series for the comparison test are the p series and the geometric series. If the series "acts like" a p series, or "acts like" a geometric series, then you may wish to use the comparison test. Remember, if $0 \le a_n \le b_n$ and

- $\sum b_n$ converges, then $\sum a_n$ converges.
- $\sum a_n$ diverges, then $\sum b_n$ diverges.

Limit Comparison Test This test works well for the type of problems that also work with the comparison test, but is somewhat easier. You still need the test series, but you don't need to work to make the terms of the series greater than or less than some known series. You only need to check the limit

$$\lim_{n\to\infty} \frac{a_n}{b_n}.$$

If it is finite and positive, then both series converge or both diverge. Since you already know about one of them, you then know about the other.

Integral Test If we are trying to determine whether $\sum a_n$ converges, and there is a function $f(x)$ with $f(n) = a_n$, then the sum converges iff

$$\int_a^\infty f(x)\,dx \text{ converges.}$$

(We assume that both the series $\{a_n\}$ and $f(x)$ are positive.) So the integral test is handy if the associated function can be integrated without too much difficulty.

Alternating Series Test To use the alternating series test, you need to verify three things: The series is alternating. (This can usually be done by inspection). The terms of the series converge to 0. (Hopefully you did this when you applied the Divergence test.) Finally, the terms of the absolute values are decreasing. The second statement does not necessarily imply the third. If this is true, then the alternating series test tells us the series converges.

Ratio Test If

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L,$$

then the series is absolutely convergent if $L < 1$ and divergent if $L > 1$. If $L = 1$, the test fails. This test works really well when a factorial is present in a_n.

Root Test If

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L,$$

then the series is absolutely convergent if $L < 1$ and divergent if $L > 1$. If $L = 1$, the test fails. This test works really well when there are powers of n in a_n.

Remember, the Integral test and the comparison tests only work when the series has non-negative terms. If you have a series where the terms are both positive and negative, then you must be able to say whether the series converges absolutely, converges conditionally, or diverges. It **is** one of these. These are mutually exclusive conditions.

Power Series

Recall that a power series is a series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n.$$

The value c is called the **center** of the power series, and the values a_n are called the coefficients.

A power series is a way to represent a function. However, the power series may have a different domain than the function does. To find the domain of the power series, (called the **interval of convergence**), we do the following:

1. Apply the ratio or root test to the power series. If the limit is 0, the power series converges everywhere and the radius of convergence is ∞. If the limit is ∞, the power series converges only at the center, and the radius of convergence is 0. Otherwise, set the limit to be less than 1, and rework the inequality so it says $|x-c| < R$. R is the radius of convergence.
2. The power series is now guaranteed to converge absolutely on $(c-R, c+R)$, and diverge on $(-\infty, c-R) \cup (c+R, \infty)$. We now test the power series at the endpoints. Plug the endpoints $c-R$ and $c+R$ into the power series and use one of the **other** 5 tests (not Ratio, not Root) to determine whether they converge. State the interval of convergence using parentheses to indicate the power series does not converge at an endpoint, and a bracket to indicate it does.

Finding sums of series

Finding a power series that represents a specific function is the next topic. The first one we learned was the geometric series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \text{ iff } x \in (-1, 1).$$

We then found the sum of several series by differentiating, integrating, multiplying by x, etc.

The Taylor series of a function is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

and can also be used to find the power series of a function.

Notice that the interval of convergence of these series is still very important. We need to know when we can trust them.

In addition to the geometric series above, the following Maclaurin series (with interval of convergence) are important:

- $\tan^{-1} x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$, $[-1, 1]$
- $\ln(1+x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$, $(-1, 1]$
- $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$, $(-\infty, \infty)$
- $(1+x)^r = \displaystyle\sum_{n=0}^{\infty} \binom{r}{n} x^n$, $(-1, 1)$ where the binomial coefficients are $\binom{r}{n} = \frac{r(r-1)\cdots(r-n+1)}{n!}$
- $\sin x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, $(-\infty, \infty)$
- $\cos x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, $(-\infty, \infty)$
- $\sinh x = \displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$, $(-\infty, \infty)$
- $\cosh x = \displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$, $(-\infty, \infty)$

If you need to construct a Maclaurin series of a function and some of the above functions are included, it is almost always easier to manipulate the Maclaurin series instead of constructing the series by scratch.

Approximating sums of series

In addition to finding whether sums of series converge or not, we also were able to find approximations to the error. There were 3 basic approximations to the error given by the Integral test, Alternating Series test, and the Taylor Series.

1. If $\sum a_k$ is convergent with sum s and $f(k) = a_k$ where f is a continuous, positive, and decreasing function for $x \geq n$, then the remainder $R_n = s - s_n = \sum_{k=n+1}^{\infty} a_k$ satisfies the inequality
$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx$$

2. If $\{a_k\}$ is a positive decreasing sequence with a limit of 0, then $\sum(-1)^k a_k$ is convergent with sum s and the remainder $R_n = s - s_n = \sum_{k=n+1}^{\infty} (-1)^k a_k$ satisfies the inequality
$$|R_n| < a_{n+1}$$

3. **Taylor's Inequality**: If $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$ is the nth Taylor polynomial of $f(x)$ centered at c, then the remainder $R_n(x) = f(x) - T_n(x)$ satisfies the inequality
$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-c|^{n+1}$$
on the interval where $|f^{(n+1)}(x)| < M$.

We use this information, when applicable, to find maximum errors when approximating a function by a Taylor polynomial as well.

Questions

Try to study the review notes and memorize any relevant equations **before** trying to work these equations. If you cannot solve a problem without the book or notes, you will not be able to solve that problem on the exam.

For problems 1 through 6, determine whether the series is absolutely convergent, conditionally convergent, or divergent.

1. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

2. $\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{n!}$

3. $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{2n+1}$

4. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)}$

5. $\sum_{n=1}^{\infty} \frac{(-1)^n (\ln n)^2}{n}$

6. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^5+3}}$

7. Show that $\frac{1}{24}$ is an upper bound on the error of $\sum_{n=1}^{\infty} \frac{1}{n^4+7}$ if the sum is approximated by the first two terms.

8. Suppose the power series $\sum_{n=2}^{\infty} a_n (x+1)^n$ has a radius of convergence $R = 5$. List all possible intervals of convergence.

9. Find the radius and interval of convergence of $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^n}$

10. Find the radius and interval of convergence of $\sum_{n=1}^{\infty} \frac{(-4)^n (x-2)^n}{3+2n}$

11. Find the radius and interval of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{3^n}$

12. Find the radius and interval of convergence of $\sum_{n=1}^{\infty} \frac{n 2^n x^n}{n^3-1}$

13. Find the radius and interval of convergence of $\sum_{n=1}^{\infty} \frac{n!\, x^n}{100^n}$

14. Find a power series representation in powers of x for the function
$$f(x) = \frac{1}{3+x}$$
with interval of convergence.

15. Find a power series representation in powers of $(x-1)$ for the function $f(x) = \frac{1}{1+x}$ and give the interval of convergence.

16. Find a power series representation in powers of $(x-1)$ for $\ln(1+x)$.

17. What is the power series representation of $\frac{x^2}{(1-x)^2}$?

18. Find the Maclaurin series for $f(x) = \ln(2-x)$ from the definition of a Maclaurin series. Find the radius of convergence.

19. Find a Taylor series for $f(x) = \cos(\pi x)$ centered at $x = 1$. Prove that the series you find represents $\cos(\pi x)$ for all x.

20. Use multiplication to find the first 4 terms of the Maclaurin series for $f(x) = e^x \cosh(2x)$.

21. Use division to find the first 3 terms of the Maclaurin series for $g(x) = \frac{x^2}{\cos x - 1}$.

22. Use the power series of $\frac{1}{\sqrt[3]{1+x}}$ to estimate $\frac{1}{\sqrt[3]{1.1}}$ correct to the nearest 0.0001. Justify that the error is less than 0.0001 using the Alternating Series Estimation Theory or Taylor's Inequality.

23. Find the sum:

    (a) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(\sqrt{3})^{2n+1}(2n+1)}$

    (b) $\sum_{n=2}^{\infty} \frac{3}{2^n n!}$

    (c) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$

    (d) $\frac{4}{2!} + \frac{8}{3!} + \frac{16}{4!} + \frac{32}{5!} + \ldots.$

24. Find the Taylor polynomial $T_3(x)$ for the function $f(x) = \arcsin x$, at $a = 0$.

25. Approximate f by a Taylor polynomial with degree n at the number a. And use Taylor's Inequality to estimate the accuracy of the approximation $f(x) \approx T_n(x)$ when x lies in the given interval.
    (a) $f(x) = \sqrt[3]{x}, \quad a = 8, \quad n = 2, \quad 7 \le x \le 9$
    (b) $f(x) = x \sin x, \quad a = 0, \quad n = 4, \quad -1 \le x \le 1$

26. Find the Taylor polynomial $T_3(x)$ for the function $f(x) = \cos x$ at the number $a = \pi/2$. And use it to estimate $\cos 80^0$ correct to five decimal places.

27. A car is moving with speed $20m/s$ and acceleration $2m/s^2$ at a given instant. Using a second-degree Taylor polynomial, estimate how far the car moves in the next second. Would it be reasonable to use this polynomial to estimate distance traveled during the next minute?

28. Show that T_n and f have the same derivatives at a up to order n.

Answers

1. Converges by the Alternating Series test. By the Integral Test, it does not converge absolutely. So it converges conditionally.

2. Diverges by the Test for Divergence.

3. Converges by the Alternating Series test. By the Limit Comparison test (with $b_n = \frac{1}{n}$), it does not converge absolutely. So it converges conditionally.

4. Converges absolutely by the Limit Comparison test (with $b_n = \frac{1}{n^2}$).

5. Converges by the Alternating Series test (Use L'Hôpital's rule). By the Integral Test, it does not converge absolutely. So it converges conditionally.

6. Converges absolutely by the Limit Comparison test (with $b_n = \frac{1}{n^{3/2}}$).

7. $R_2 = \sum_{n=3}^{\infty} \frac{1}{n^4+7} \leq \sum_{n=3}^{\infty} \frac{1}{n^4} \leq \int_2^{\infty} \frac{1}{x^4}\, dx = \frac{1}{24}$

8. (-6,4), (-6,4], [-6,4), [-6,4]

9. $\left|\frac{a_{n+1}}{a_n}\right| \to 0$ for all x: $R = \infty$, $I = (-\infty, \infty)$

10. $\left|\frac{a_{n+1}}{a_n}\right| \to 4|x-2| < 1$: $R = \frac{1}{4}$, $I = (\frac{7}{4}, \frac{9}{4}]$

11. $\left|\frac{a_{n+1}}{a_n}\right| \to \frac{|x|}{3} < 1$: $R = 3$, $I = (-3, 3)$

12. $\left|\frac{a_{n+1}}{a_n}\right| \to 2|x| < 1$: $R = \frac{1}{2}$, $I = [-\frac{1}{2}, \frac{1}{2}]$

13. $\left|\frac{a_{n+1}}{a_n}\right| \to \infty$ for all x: $R = 0$, $I = \{0\}$

14. $\frac{1}{3+x} = \frac{1}{3}\left(\frac{1}{1-(-\frac{x}{3})}\right) = \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n$
$= \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^{n+1}}$ for $x \in (-3, 3)$

15. $\frac{1}{1+x} = \frac{1}{2+(x-1)} = \frac{1}{2} \cdot \frac{1}{1-\left(\frac{-(x-1)}{2}\right)}$
$= \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}(x-1)^n$ for $x \in (-1, 3)$

16. Integrate the previous solution to get
$\ln(1+x) = C + \frac{1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n(n+1)}(x-1)^{n+1}$: $(C = \ln 2)$

17. $\frac{x^2}{(1-x)^2} = x^2 \frac{d}{dx}\left(\frac{1}{1-x}\right) = x^2 \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right)$
$= x^2 \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=1}^{\infty} n x^{n+1}$

18. $\ln(2-x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \ln 2 + \sum_{n=1}^{\infty} \frac{-x^n}{2^n n}$: $(R = 2)$

19. $\cos(\pi x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\pi^{2n}(x-1)^{2n}}{(2n)!}$

$|R_n(x)| \leq \frac{\pi^{n+1}|x-1|^{n+1}}{(n+1)!} \to 0$ for all x

20. $e^x \cosh 2x = (1 + x + \frac{x^2}{2!} + \cdots\ldots)(1 + \frac{(2x)^2}{2!} + \cdots) =$
$1 + x + \frac{5}{2}x^2 + \frac{13}{6}x^3 + \cdots$

21. $\frac{x^2}{\cos x - 1} = \frac{x^2}{-\frac{x^2}{2!} + \frac{x^4}{4!} - \ldots} = -2 - \frac{x^2}{6} - \frac{x^4}{120} + \cdots$

22. $\frac{1}{\sqrt[3]{1+x}} = 1 - \frac{x}{3} + \frac{2x^2}{9} - \frac{14x^3}{81} + \cdots$
Thus, $\frac{1}{\sqrt[3]{1.1}} \approx 1 - \frac{1}{30} + \frac{1}{450}$. Since the series is alternating the error for this sum is less than the size of the next term, which is $\frac{7}{40500}$, which is less than 0.001.

23. Find the sum:

    (a) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(\sqrt{3})^{2n+1}(2n+1)} = \tan^{-1}(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$

    (b) $\sum_{n=0}^{\infty} \frac{3}{2^n n!} = 3\sqrt{e}$

    (c) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = \frac{\sin x}{x}$

    (d) $\frac{4}{2!} + \frac{8}{3!} + \frac{16}{4!} + \frac{32}{5!} + \ldots. = e^2 - 3$

24. $T_3(x) = x + \frac{1}{6}x^3$

25. (a) $2 + \frac{x-8}{12} - \frac{(x-8)^2}{288}$: $|R_2| \leq \frac{f^{(3)}(7)\cdot 1^3}{3!} \approx 0.00034$
(b) $x^2 - \frac{x^4}{6}$: $|R_4| \leq \frac{f^{(5)}(1)\cdot 1^5}{5!} \approx 0.0396$

26. $-(x - \frac{\pi}{2}) + \frac{1}{6}(x - \frac{\pi}{2})^2$: $\cos 80° = \cos\frac{4\pi}{9} \approx 0.174$

27. $T_2(x) = s(0) + s'(0)x + \frac{s''(0)}{2}x^2 = 20x + x^2$
$T_2(1) = 21$ m: No

28. Prove by mathematical induction or directly consider the k^{th} derivative of the polynomial T_n.