Math 113 Midterm 1 Review Sheet

Remember: Calculators are not allowed on the exam.

Some of the integrals appearing in the chapter 6 problems may require the techniques of chapter 7 to evaluate.

6.1: Areas between Curves

Here are the important steps to keep in mind:

- Begin by sketching the curves together in one coordinate plane. This is the most important step; be sure you can do it without using a calculator.
- Choose which variable to integrate with respect to. If the curves give y as a function of x (e.g., $y = x^3 - x + 1$), you will want to integrate with respect to x, while if the curves give x as a function of y (e.g., $x = e^y + y$), you will want integrate with respect to y. In some cases it may be possible to solve for either variable (e.g., $2x + 3y = 5$).
- Find the intersection points of the curves. These will determine the bounds of the integral(s). If you are integrating with respect to x, then you are interested in the x coordinates of the intersection points, while if you are integrating with respect to y, then you are interested in the y coordinates of the intersection points.
- Keep in mind that the area of the region is found by splitting it up into thin approximately rectangular pieces; the width of these rectangles become the dx or dy in the integral, while the height (or length) of the rectangles are found by subtracting the y-coordinate of the curve on top from the y-coordinate of the curve on bottom (or, if you are integrating with respect to y, by subtracting the x-coordinate of the curve on the right from the x-coordinate of the curve on the left).
- It may be necessary to split the region into two or more pieces. But this can sometimes be avoided by integrating with respect to the other variable.

Find the area of the region enclosed by the given curves.

1. $y = x^3 - x + 1$, $y = 1$, $2 \leq x \leq 3$
2. $y = x^2 - 1$, $y = 2 - x - x^2$
3. $x^2 + (y - 1)^2 = 1$, $y = x$
4. $y = x^2$, $y = 3x^2$, $2x + y = 1$, $x \geq 0$
5. $y = \sqrt{x^2 + 1}$, $x = 0$, $x = 1$, $y = -1$

6.2 & 6.3: Volumes

The guidelines above also apply to these problems. For solids of revolution, you will have to decide whether to use the method of slicing or cylindrical shells:

- If the axis along which you are integrating is parallel to the axis of rotation, use the method of slicing. If the axis along which you are integrating is perpendicular to the axis of rotation, use the method of cylindrical shells.
- Keep in mind that the volume of the solid is found by splitting up the region into thin approximately rectangular pieces and considering the volume swept out by each of these rectangles when they are revolved about the appropriate axis. Each small rectangle (which in the limit becomes simply a line segment) will sweep out either a washer or a cylindrical shell. Visualizing this process will make it easy to determine which method is applicable for a given problem.
- The area of a washer is $\pi(r_{out}^2 - r_{in}^2)$, where r_{out} is the radius of the outer circle and r_{in} is the radius of the inner circle; r_{out} and r_{in} are found by measuring the distance from the two curves to the axis of rotation.
- The area of a cylindrical shell is $2\pi rh$, where r is the radius (distance to the axis of rotation) and h is the height of the cylinder (found by subtracting: the top curve minus the bottom curve, or the right curve minus the left curve, depending on which variable we are integrating with respect to).
- Some solids do not result from revolving a region about an axis. For such solids, the method of cylindrical shells is not possible. The volume must be found by slicing. To do this, choose an axis along which to integrate; identify the type of cross sections perpendicular to this axis (whether they be triangles, squares, disks, etc.); and calculate the area of such cross sections. The volume is the integral of these areas.

Find the volume of the solid obtained by revolving the region bounded by the given curves about the given axis.

6. $y = 4 - x^2, y = 0$; about the x-axis
7. $x = -y - 1, x = e^y, 0 \leq y \leq 1$; about the x-axis
8. $y = \sin x + 1, y = -1, x = 0, x = \frac{\pi}{4}$; about the line $x = -1$
9. $y = \sec x, y = 0, x = 0, x = \frac{\pi}{3}$; about the line $y = 3$
10. $x = \sqrt{y} + y, x = 0, y = 0, y = 1$; about the line $y = -1$

Find the volume of the solid described.

11. The base of the solid is the unit disk $x^2 + y^2 \leq 1$. Cross sections perpendicular to the x-axis are squares.
12. The base of the solid is the region enclosed by the curves $x + y = 1$, $x + y = -1$, $x - y = 1$, and $x - y = -1$. Cross sections perpendicular to the x-axis are equilateral triangles.

6.4: Work

The work done by a force on an object is given by $W = Fd$, where F is the magnitude of the force and d is the displacement of the object in the direction of the force. In each problem, it is important to first **establish a coordinate system**: decide where to place the origin and in which direction to point the positive axis.

- If a varying force acts on an object, you should divide the axis of the object's movement into small intervals and calculate the work done by the force on the object as it moves over the length of each small interval.
- If the object consists of parts, each of which is to be displaced by a different amount, divide the object into small parts and calculate the work done by the force in moving each part of the object to its final location.

13. A chain weighing 3 lb/ft is used to lift a 500 lb object a height of 20 ft, to the level of the top of the chain. Find the work done.
14. A particle is moved along the x-axis from the origin a distance of 5 meters by a force which varies depending on the position of the particle. When the particle is at x, the force is $(2x+1)/(x+1)$ newtons in the positive direction. Find the work done by the force on the particle. [Hint: Use long division to simplify the integrand.]
15. A force of 20N is required to hold a spring stretched 40cm, while a force of 30N is required to hold it stretched 45cm. How much work is required to stretch the spring from 50cm to 60cm? [Hint: Find the natural length of the spring.]
16. The great pyramid of Giza consists of approximately 2 million stones, each weighing 1.5 tons (3000 lb). The pyramid is 450 ft high with a square base measuring 750 ft. Find the work required to lift the stones into place from ground level. [Hint: To simplify your calculations, work the problem symbolically; only plug in the given numbers at the last step.]

7.1: Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

- Integration by parts is most often useful when integrating a function of the form $x^n e^x$, $x^n \sin x$, $x^n \cos x$, $x^n \ln x$. If possible, you want to choose u to be a function that becomes simpler when differentiated, and dv to be a function that can be readily integrated. This usually means you should choose $u = x^n$. (But in the case $x^n \ln x$, choose $u = \ln x$).
- Integration by parts is also useful for integrating inverse functions such as $\sin^{-1} x$, $\tan^{-1} x$, $\ln x$ or functions involving these as factors. In this case, you should choose $u = \sin^{-1} x$, $u = \tan^{-1} x$, or $u = \ln x$ accordingly, even if there are no other factors in the integrand (i.e., you can set $dv = dx$).

Evaluate the following integrals.

17. $\int x \cos x\,dx$

18. $\int_0^{\frac{\pi}{2}} x \sin x \; dx$

19. $\int_0^1 x^2 e^x \; dx$

20. $\int_0^1 \sin^{-1} x \; dx$

21. $\int 2x \tan^{-1} x \; dx$

22. $\int \frac{\ln x}{x^2} \; dx$

23. $\int e^x \cos x$

24. $\int_{\sqrt{\frac{\pi}{2}}}^{\sqrt{\pi}} x^3 \sin(x^2) \; dx$ [Hint: First use a u-substitution]

7.2: Trigonometric Integrals-sin and cos only

- For $\int \sin^m x \cos^n x \; dx$:
  If n is odd, save one $\cos x$ and convert the rest to sin using $\cos^2 x = 1 - \sin^2 x$
  If m is odd, save one $\sin x$ and convert the rest to cos using $\sin^2 x = 1 - \cos^2 x$
  If both m and n are even, use the identities $\sin^2 x = \frac{1-\cos 2x}{2}$ and $\cos^2 x = \frac{1+\cos 2x}{2}$.
  If the periods of sin and cos are different, use the product formulas.

Evaluate the following integrals:

25. $\int_0^{\frac{\pi}{6}} \sin^2 x \cos^3 x \; dx$

26. $\int_0^{\frac{\pi}{3}} \cos^4 x \; dx$

27. $\int_0^{\frac{\pi}{4}} \sin^5(2x) \cos^4(2x) \; dx$

28. $\int \sin(2x) \cos(5x) \; dx$

Answers

1. $\frac{55}{4}$
2. $\frac{125}{24}$
3. $\frac{\pi}{4} - \frac{1}{2}$
4. $\frac{4}{3}\sqrt{2} - \frac{50}{27}$
5. $1 + \frac{1}{2}\sqrt{2} + \frac{1}{2}\ln(\sqrt{2}+1)$
6. $\frac{512}{15}\pi$
7. $\frac{11}{3}\pi$
8. $\pi(\frac{\pi^2}{8} - \frac{\sqrt{2}}{4}\pi + \pi + 2)$
9. $\pi(6\ln(2+\sqrt{3}) - \sqrt{3})$
10. $\frac{19}{5}\pi$
11. 4
12. $\frac{2}{3}\sqrt{3}$
13. 10600 ft-ln
14. $10 - \ln 6$
15. 5 J
16. 675,000,000 ft-lb
17. $x\sin x + \cos x + C$
18. 1
19. $e - 2$
20. $\frac{\pi}{2} - 1$
21. $x^2\tan^{-1}x - x + \tan^{-1}x + C$
22. $\frac{1}{3}x^3\ln x - \frac{1}{9}x^3 + C$
23. $\frac{1}{2}e^x(\sin x + \cos x) + C$
24. $\frac{1}{2}(\pi - 1)$
25. $\frac{17}{480}$
26. $\frac{\pi}{8} + \frac{7}{64}\sqrt{3}$
27. $\dfrac{4}{315}$
28. $-1/14\,\cos(7\,x) + 1/6\,\cos(3\,x) + C$