Welcome to Math 343. We meet only 21 times (see the calendar in this document, which you should keep handy). Nevertheless, we cover all the essentials in the course book (Roger Baker, Linear Algebra, Rinton Press 2001). If you want to know about linear algebra in economics, engineering, computer graphics and other interesting and important applications, browse in David Lay, Linear Algebra and Its Applications in the library. In real world applications, you will use a combination of theory and computer programs. In this course, we just want to learn the basic theory, and it is not hard to do all the required calculation by hand. I shall stick to one nice application (Markov chains).

I should emphasize that accurate arithmetic in your calculations is a crucial part of your course. I can’t give much credit if you do a calculation wrongly, because you have not done the required task. In most cases there are neat ways to check your work and I will, repeatedly, show you these. Once more, for emphasis: if you want a good grade you have to earn it by checking every calculation you make. It should help that the exams are take-home, and you don’t have to race the clock when doing your checking.

Regarding exams, you may not discuss them with anyone (your spouse, parent, classmate, etc.). If you do, you are in breach of the honor code. But you can come and talk to me and I can help, in the sense of doing relevant problems with you. My office is in 282 TMCB.

I want to go over a good many homework questions in class, so if you hand in homework late, there is no credit if I already did this. You also must have a really good excuse to get permission to hand in homework late. Obviously every class and every homework is important with our compressed schedule.

Here are a few more important facts.

(i) Homework counts for 30%, the take-home midterm for 35%, and the take-home final for 35% of your grade. Homework questions are out of 10 and all count equally.

(ii) I don’t grade on a curve. I would be only too pleased to give an A to everyone if your work is good.

(iii) You have to get a course total of 90% for an A, 80% for a B and 70% for a C.

(iv) Homework is usually due on Wednesday (covering Monday’s lecture) and Monday (covering Wednesday’s and Friday’s lectures). The complete set of homework questions (with due dates) is attached. If you lose them, go to my faculty web page at www.math.byu.edu/, where this document can be found. I intent to give out printed solutions once I have gone over homework in class, but it is important to listen to the class explanations of solutions to extract as much information as possible.

(v) There is a holiday on Monday July 24.

(vi) I can be found in my office before the lecture (11-12) and after (2-3) on MWF. On Tuesday you can come between 11 and 2 to ask questions. I can provide expert help with homework by doing similar questions. With this service available, you should not
have trouble turning in good answers! You can also work in groups on homework, but you must write up your work **individually** once you have roughed out solutions in a group. (Please come and see me occasionally even if you don’t need help. I like to get to know my students.) Duokui’s office hours are Thursdays from 1:00–3:00.

You **must** keep all homework and exams once they are returned, since sometimes I need to take a second look to adjust scores. It is a very good idea to make a final draft of the exams before turning them in. Messy or cramped work is unlikely to be accurate or clearly explained.

Please turn off your cell phones in class.

Here are some standard department policies.

*Preventing Sexual Harassment*: BYU’s policy against sexual harassment extends not only to employees of the university but to students as well. If you encounter unlawful sexual harassment, gender-based discrimination, or other inappropriate behavior, please talk to your professor; contact the Equal Employment Office at 422-5895; or contact the Honor Code Office at 422-2847.

*Students with Disabilities*: BYU is committed to providing reasonable accommodation to qualified persons with disabilities. If you have any disability that may adversely affect your success in this course, please contact the University Accessibility Center at 422-2767. Services deemed appropriate will be coordinated with the student and instructor by that office.

**Math 343 - Summer 2006 - Calendar**

(In listing sections covered, 1.2 means Chapter 1, Section 2, for example.)

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The midterm covers Chapters 1–4 and the final, Chapters 5–8.
1. Find the equation of the straight line through \((3, -1)\) perpendicular to the direction \((7, 8)\).

2. Find the sum of the three vectors \((6, 2, 1), (4, -1, 2)\) and \((0, 1, 5)\).

3. Among the vectors \((2, 1, 5), (-2, -1, 1), (0, 3, 3)\), find all the pairs of perpendicular vectors.

4. Find the distance from the point \((4, -2)\) to the straight line
   \[5x_1 - 9x_2 = 2.\]

5. Among the vectors \((6, 2, 2), (3, 4, -1), (0, 2, 5)\), which pair has the smallest angle between them? (You need not compute the three angles.)

6. Find the point on the line
   \[4x_1 - x_2 = 6\]
   obtained by dropping a perpendicular from \((1, 9)\).

7. Find the point that is two-fifths of the way along the line segment starting at \((7, 1, 1)\) and ending at \((6, 3, -1)\).
1. Find the parametric equation of the straight line through $(3, 1, 2, -1)$ and $(4, 6, 5, 7)$.

2. Find the equation of the plane passing through $(1, 7, -3), (-2, 1, 0)$ and $(0, -2, -1)$.

3. Write the plane with equation $x_1 + 6x_2 - 4x_3 = 12$ in parametric form.

4. A normal to the plane with equation $3x_1 - x_2 - 9x_3 = 38$ passes through the point $(2, -1, -2)$. Where does this normal meet the plane?

5. Show that four distinct points $a, b, c, d$ all lie in a single plane if there is a solution $(x_1, x_2)$ of the vector equation $a + x_1(b - a) = c + x_2(d - c)$. 

6. Find the projection of $(6, 2, -1)$ on $(4, 4, 3)$.

7. Check that the Cauchy-Schwarz inequality is satisfied by the pair of vectors $(1, 1, 2, 3)$ and $(2, 3, 3, 3)$.

8. Using the formula on page 32, find the distance from the point $(3, -1, 2, -1)$ to the hyperplane $x_1 - x_2 + x_3 + 2x_4 = 3$.

9. Simplify the linear system

\[
\begin{align*}
2x_1 & - x_2 + 4x_3 = 12 \\
5x_1 & + x_2 + x_3 = 10 \\
-x_1 + 4x_2 & + x_3 = 10
\end{align*}
\]

using row reduction. Show that there is a unique solution. Which of the matrices you obtained in your working are in echelon form, and which are in reduced echelon form?

10. Row reduce the following matrices to reduced echelon form. Show your row operations as in Example 4, page 43.

\[
\begin{bmatrix}
2 & 3 \\
1 & 2
\end{bmatrix}, \begin{bmatrix}
3 & 2 & 1 \\
7 & 2 & 5/2
\end{bmatrix}, \begin{bmatrix}
12 & 4 & 4 & 2 \\
2 & 0 & 10 & 1
\end{bmatrix}, \begin{bmatrix}
4 & 2 & 4 & 1 \\
3 & 2 & 2 & 1 \\
1 & 2 & -2 & 0 \\
6 & 2 & 2 & 2
\end{bmatrix}
\]

11. Find the pivot columns of the matrices in question 10.
1–5. Solve exercises 4.4, 4.5, 4.6, 4.7, 4.8 of Chapter 2. (The task is explained on page 55 before 4.1.)

6. Let $x_1, x_2, x_3, x_4$ be real numbers such that $x_1$ is the average of $x_2, x_3, x_4$, that is

$$x_1 = \frac{x_2 + x_3 + x_4}{3},$$

$x_2$ is the average of $x_1, x_2, x_3$, and

$x_3$ is the average of $x_1, x_2, x_4$.

Show that $x_1 = x_2 = x_3 = x_4$.

*Hint:* This amounts to solving a homogeneous system.
1. Find the linear span of the vectors \( \mathbf{v}_1 = (1, 2, 7), \mathbf{v}_2 = (2, 1, 1) \).

Determine whether the column vectors of \( A \) form a linearly independent set in each of the following four cases.

2. \( A = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 0 & 1 & 2 & 0 \\ 2 & 0 & 3 & 6 \\ 1 & 1 & 3 & 1 \end{bmatrix} \).

3. \( A = \begin{bmatrix} 1 & 3 & 5 \\ 4 & -1 & -2 \end{bmatrix} \).

4. \( A = \begin{bmatrix} 1 & -1 & 3 & 2 & 1 \\ 2 & 4 & 6 & 1 & 0 \\ 3 & 3 & 9 & 3 & 1 \\ 2 & 0 & 1 & 1 & 2 \end{bmatrix} \).

5. \( A = \begin{bmatrix} 2 & 3 & 5 \\ 7 & 11 & 13 \\ 17 & 19 & 23 \end{bmatrix} \).

6–9. Find a basis of \( \text{Nul} \, A \), as in questions 2–5. Give the nullity of \( A \) in each case.

10. Let \( A \) be an \( m \times n \) matrix with \( n > m \). Show that the nullity of \( A \) is at least \( n - m \).

11. Let the matrix \( B \) be obtained from the matrix \( A \) by discarding the last column of \( A \). Is it true that the nullity of \( B \) is \( \leq \) nullity of \( A \)? Can we have nullity \( B < \) nullity \( A \)?

    \text{Hint: } Let \( A \) be \( m \times n \) with \( k \) pivot columns. How many pivot columns could \( B \) have?

12. Let \( \mathbf{v}_1 = (1, 1, 0), \mathbf{v}_2 = (1, 0, 1) \). Find \( \mathbf{v}_3 \) so that \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) is a basis of \( \mathbb{R}^3 \).

13. Let \( V \) be the hyperplane in \( \mathbb{R}^4 \) with equation

    \[ x_1 + 2x_2 + 3x_3 - x_4 = 0. \]

Let \( \mathbf{v}_1 = (1, 2, 1, 8) \) and \( \mathbf{v}_2 = (0, 1, 2, 8) \). Find \( \mathbf{v}_3 \) so that \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) is a basis of \( V \).

    \text{Hint: } Just pick a ‘reasonable’ \( \mathbf{v}_3 \) and make sure \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) is an independent set.
1–4. For each of the following matrices $A$, find the rank and nullity of $A$.

\[
\begin{bmatrix}
-1 & 2 \\
3 & 4 \\
\end{bmatrix},
\begin{bmatrix}
-1 & 3 & 5 \\
2 & -6 & -10 \\
\end{bmatrix},
\begin{bmatrix}
2 & 1 & 3 \\
18 & 12 & 4 \\
-13 & -8 & -8 \\
\end{bmatrix},
\begin{bmatrix}
2 & 0 & 1 & -1 \\
2 & 0 & -1 & 1 \\
0 & 1 & 1 & -2 \\
1 & 2 & 0 & 1 \\
\end{bmatrix}.
\]

5–8. For the matrices in question 1, find a ‘hidden relation’ between the columns of $A$ (if there is one).

9. Let $v_1 = (6, 4, -1, -2)$, $v_2 = (5, 7, 1, -3)$, $v_3 = (0, 1, -2, 2)$. Find the equation of the hyperplane $H$ that contains $v_1$, $v_2$ and $v_3$ and $0$. Using the idea of dimension of a subspace, explain why it is true that

\[H = \text{Span}\{v_1, v_2, v_3\}.\]
Math 343 - Summer 2006


1. Let \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^4 \),

\[
T(x_1, x_2, x_3) = (x_1 + x_2 + 7x_3, 2x_1 - x_2 + 3x_3, 6x_1 + x_3, x_2 - x_3).
\]

Find \( Ta_1 \), \( Ta_2 \) where \( a_1 = (1, 1, 1) \), \( a_2 = (3, 2, 4) \). Find \( T(a_1 + a_2) \) with as little calculation as possible.

2. Find the matrix of the linear mapping \( T \) in question 1.

3. Let \( T \) be an anticlockwise rotation, through \( \pi/4 \), of \( \mathbb{R}^2 \). Find the image \( Tx \) when \( x = (3, 1) \).

4. Let \( T \) be the mapping in question 1. Is \( T \) one-to-one? Is \( T \) onto?

5. Let \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be a rotation through angle \( a \). Using the matrix formula for \( T \), show that \( |T x| = |x| \).

6. Which of the three possible pairs of matrices, taken from the following three, commute?

\[
A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 2 \\ 3 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}.
\]

7. Let \( A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \). Compute \( A^2 \) and \( A^3 \). Guess a formula for \( A^n \) (\( n = 1, 2, \ldots \)) and prove it by induction.

8. By arguing as on page 94, compute the inverse of the mapping \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \),

\[
Tx = (x_1 + x_2 - 9x_3, \ x_2 - x_3, \ x_1 + 4x_3).
\]

What is the inverse of the matrix of \( T \)?

9. If \( T \) is a rotation of \( \mathbb{R}^3 \) about the vertical axis \( e_3 \), through angle \( a \), what is \( Tx \)? Is \( T \) linear?

10. Let \( A, B, C \) be invertible \( 10 \times 10 \) matrices. Write down the inverse of

\[
A^2B^2CAC^{-1}B^3A^2C^4.
\]

11. Let

\[
A = \begin{bmatrix} 2 & 3 & 0 & 1 \\ 1 & 3 & 0 & 2 \\ 2 & 3 & 0 & 0 \\ 1 & 2 & 1 & 3 \end{bmatrix}.
\]

Find the inverse of \( A \). Verify that your answer has \( AB = I \).
1. Let $A = \begin{bmatrix} 3 & 5 & 6 \\ 1 & 0 & 1 \\ 2 & 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 0 & 2 \\ 1 & 0 & 3 \end{bmatrix}$. Compute $AB$, $(AB)^t$. Compute $B^t A^t$ without using $(AB)^t = B^t A^t$. Check that the last equation is true.

2. Let $A$ be the matrix of a rotation of $\mathbb{R}^2$. Show that $A^t$ is also the matrix of a rotation.

3. Show that the matrix

$$P = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ -2/3 & 2/3 & -1/3 \end{bmatrix}$$

is orthogonal

4. Let

$$Q = \begin{bmatrix} a & b & b \\ b & a & -b \\ -b & b & -a \end{bmatrix}$$

be orthogonal. Show that $Q$ is $\pm I$ or $\pm P$, where $P$ is the matrix in question 3.
1. Determine the parity of the following permutations.
   (i) 1, 2, 5, 6, 3, 7, 4.
   (ii) 1, 4, 6, 2, 5, 3.
   (iii) 8, 7, 6, 5, 4, 3, 2, 1.
   (iv) 2, 3, 5, 7, 11, 1, 4, 6, 8, 9, 10.

2. Let
   \[ A = \begin{bmatrix}
   2 & 3 & 0 & 1 \\
   1 & 3 & 0 & 2 \\
   2 & 3 & 1 & 0 \\
   1 & 2 & 0 & -3
   \end{bmatrix}. \]

   Find the determinant of \( A \).

3. Find the determinant of
   \[ \begin{bmatrix}
   1 & 2 & 0 & 0 \\
   3 & 4 & 0 & 0 \\
   10 & 20 & 5 & 6 \\
   30 & 40 & 7 & 8
   \end{bmatrix}. \]

4. Find the adjoint of
   \[ A = \begin{bmatrix}
   a & b & a-2b \\
   c & d & c-2d \\
   e & f & e-2f
   \end{bmatrix}. \]

   Work out \( A \ adj \ A \). Was this answer predictable?

5. Find the adjoint of
   \[ A = \begin{bmatrix}
   a & 0 & 0 \\
   0 & b & c \\
   0 & d & e
   \end{bmatrix}. \]

   Write down \( A \ adj \ A \), and show that \( A \ adj \ A = 0 \) exactly when \( abe = acd \).

6. Solve the system
   \[
   \begin{align*}
   12x_1 + 12x_2 + 2x_3 &= 64 \\
   x_1 + 2x_2 - x_3 &= 4 \\
   -x_1 + 4x_2 - x_3 &= -2
   \end{align*}
   \]

   using Cramer’s rule. (The answers are whole numbers.)
1. Find the adjoint of 
\[ A = \begin{bmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{bmatrix} . \]

Work out \( A \ adj A \). Is your answer consistent with Example 16 of Chapter 5?

2. Find the volume of the parallelepiped with vertices \( a, b, c \) adjacent to \( 0 \), where
\[ a = (1, 5, 7), \ b = (0, 6, 2), \ c = (1, 3, 3). \]

3. Let \( b_1, b_2 \) with real numbers with \( b_1^2 + b_2^2 = 1 \). Find the characteristic polynomial of the projection \( S \) on page 88. Can you explain why the polynomial does not depend on \( b_1 \) and \( b_2 \)?

For each of the following matrices \( A \), find \( P \) and diagonal \( D \) such that \( A = PDP^{-1} \).

4. \( A = \begin{bmatrix} 2 & 5 \\ 0 & 6 \end{bmatrix} \).

5. \( A = \begin{bmatrix} 3 & 1 & 0 \\ 5 & -1 & 0 \\ 6 & 1 & -1 \end{bmatrix} . \)

6. \( A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 3 \end{bmatrix} \).

7. Let \( T \) and \( S \) be linear mappings of \( \mathbb{R}^3 \) onto \( \mathbb{R}^3 \). Suppose that \( Sa_1 = c_1a_1, \ Sa_2 = c_2a_2 \) and \( Sa_3 = c_3a_3 \). Suppose that \( Ta_1 = d_1a_1, \ Ta_2 = d_2a_2 \) and \( Ta_3 = d_3a_3 \). If \( a_1, a_2, a_3 \) is a linearly independent set, show that \( TS \) is diagonalizable.

\textit{Hint:} What is \( TSa_1 \)?

8. Let \( B \) be a diagonalizable matrix. Show that \( B^n \) is diagonalizable for every \( n = 1, 2, 3, \ldots \).

\textit{Hint:} Let \( Ba = ca \). What is \( B^n a \)?

1. Let $A$ be the transition matrix

\[ A = \begin{bmatrix} 0.1 & 0.6 & 0.3 \\ 0.2 & 0.2 & 0.3 \\ 0.7 & 0.2 & 0.4 \end{bmatrix}. \]

Show that there is a steady state vector $q$ that does not depend on $q_0$, and find $q$.

2. Let $A$ be the transition matrix

\[ A = \begin{bmatrix} 0.4 & 0.1 & 0.5 \\ 0.2 & 0.3 & 0 \\ 0.4 & 0.6 & 0.5 \end{bmatrix}. \]

Show that there is a steady state vector $q$ that does not depend on $q_0$, and find $q$.

3. Let $Ax = cx$ where $A = [p_{ij}]$ is a transition matrix and $c$ is not 1. Show that the coordinates of $x$ add up to 0,

\[ x_1 + \cdots + x_n = 0. \]

*Hint:* show that

\[ p_{11}x_1 + \cdots + p_{1n}x_n = cx_1 \\
p_{21}x_1 + \cdots + p_{2n}x_n = cx_2 \\
\vdots \\
p_{n1}x_1 + \cdots + p_{nn}x_n = cx_n. \]

If you add all these equations together, then $x_1 + \cdots + x_n$ is a factor on both sides of your new equation.
Homework 11. Sections 7.2 and 7.4. Due in class Wednesday, 9 August.

For each of the following linearly independent sets, find an orthogonal basis of the linear span of the set. Aim for the simplest looking basis. Check orthogonality.

1. $w_1 = (1, -2, 4), w_2 = (2, 1, 3)$.

2. $w_1 = (1, 0, 2, -1), w_2 = (-1, 1, 1, -1), w_3 = (1, -1, 0, 2)$.

For each of the following symmetric matrices $A$, find an orthogonal matrix $P$ such that $A = PDP^t$, $D$ diagonal.

3. $A = \begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$.

4. $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 2 & 0 \end{bmatrix}$.

5. $A = \begin{bmatrix} -5 & 2 & 2 \\ 2 & -5 & 2 \\ 2 & 2 & -5 \end{bmatrix}$.