Math 314 Lecture #23
§15.5: Applications of Double Integrals

Outcome A: Use double integration to recover mass from the density.

Suppose a lamina occupies a region \(D\) of the \(xy\)-plane, and is density (mass per unit area) is given by a function \(\rho(x,y)\) continuous on \(D\).

The density of the lamina at a point \((x,y)\) in \(D\) is given by
\[
\lim_{\Delta A \to 0} \frac{\Delta m}{\Delta A}
\]
where \(\Delta m\) is the mass of the rectangles \(R\) containing the point \((x,y)\) whose area \(\Delta A\) is shrinking to 0.

The mass \(m\) of the lamina is recovered from its density through the double integral:
\[
m = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \rho(x_{ij}^{*}, y_{ij}^{*}) \Delta A = \iint_{D} \rho(x,y) \, dA.
\]

Example. Find the mass of the lamina whose shape is the triangular region \(D\) enclosed by the lines \(x = 0\), \(y = x\), and \(2x + y = 6\), and whose density is \(\rho(x,y) = x + y\). Here is a picture of the region \(D\).

The region \(D\) is of both types, but is easier to render it as of type I, namely
\[
D = \{(x,y) : 0 \leq x \leq 2, x \leq y \leq 6 - 2x\}.
\]

The mass of the lamina is
\[
\iint_{D} \rho(x,y) \, dA = \int_{0}^{2} \int_{x}^{6-2x} (x+y) \, dy \, dx = \int_{0}^{2} \left[ xy + \frac{y^2}{2} \right]_{y=x}^{y=6-2x} \, dx
\]
\[
= \int_{0}^{2} \left[ x(6-2x) + \frac{(6-2x)^2}{2} - x^2 - \frac{x^2}{2} \right] \, dx
\]
\[
= \int_{0}^{2} \left[ 6x - \frac{7x^2}{2} + \frac{36 - 24x + 4x^2}{2} \right] \, dx
\]
\[
= \int_{0}^{2} \left[ 18 - 6x - \frac{3x^2}{2} \right] \, dx = \left[ 18x - 3x^2 - \frac{x^3}{2} \right]_{0}^{2} = 36 - 12 - \frac{8}{2} = 20.
\]
Outcome B: Use double integration to compute moments and center of mass of lamina.

Recall that the moment of a particle about an axis is the product of its mass and its directed distance from the axis.

The **moment** of a lamina occupying a region \( D \) with density \( \rho(x, y) \) **about the** \( x \)-axis is

\[
M_x = \iiint_D y \rho(x, y) \, dA.
\]

The **moment** of the lamina **about the** \( y \)-axis is

\[
M_y = \iiint_D x \rho(x, y) \, dA.
\]

The **center of mass** of the lamina is the point \((\bar{x}, \bar{y})\) where

\[
\bar{x} = \frac{M_y}{m}, \quad \bar{y} = \frac{M_x}{m}
\]

where \( m \) is the mass of the lamina; the center of mass is the point on which the lamina would balance perfectly.

When the density is uniform, i.e., \( \rho(x, y) \) is a constant, the center of mass is the geometric center. Where is the geometric center of North America?

**Example (continued).** The moment about the \( x \)-axis of the lamina occupying the region \( D \) enclosed by \( x = 0, \ y = x, \) and \( 2x + y = 6, \) with density \( \rho(x, y) = x + y \) is

\[
M_x = \iiint_D y \rho(x, y) \, dA = \int_0^2 \int_x^{6-2x} (x + y) \rho(x, y) \, dA = \int_0^2 \int_x^{6-2x} (xy + y^2) \, dA
\]

\[
= \int_0^2 \left[ \frac{xy^2}{2} + \frac{y^3}{3} \right]_{y=x}^{y=6-2x} \, dx
\]

\[
= \int_0^2 \left[ \frac{x(6-2x)^2}{2} + \frac{(6-2x)^3}{3} - \frac{x^3}{2} - \frac{x^3}{3} \right] \, dx
\]

\[
= \int_0^2 \left[ \frac{x(36-24x+4x^2)}{2} - \frac{5x^3}{6} + \frac{(6-2x)^3}{3} \right] \, dx
\]

\[
= \int_0^2 \left[ 18x - 12x^2 + \frac{7x^3}{6} + \frac{(6-2x)^3}{3} \right] \, dx
\]

\[
= \left[ 9x^2 - 4x^3 + \frac{7x^4}{24} - \frac{(6-2x)^4}{24} \right]_0^2
\]

\[
= 36 - 32 + \frac{7(16)}{24} - \frac{16}{24} + \frac{6^4}{24} = 62.
\]

The \( y \)-component of the center of mass is

\[
\bar{y} = \frac{M_x}{m} = \frac{62}{20} = \frac{31}{10}.
\]
The moment of the lamina about the \( y \)-axis is

\[
M_y = \iint_D x \rho(x, y) \, dA = \int_0^2 \int_x^{6-2x} (x^2 + xy) \, dy \, dx
\]

\[
= \int_0^2 \left[ x^2 y + \frac{xy^2}{2} \right]_{y=x}^{y=6-2x} \, dx
\]

\[
= \int_0^2 \left[ x^2(6 - 2x) + \frac{x(6 - 2x)^2}{2} - x^3 - \frac{x^3}{2} \right] \, dx
\]

\[
= \int_0^2 \left[ 6x^2 - 2x^3 + \frac{x(36 - 24x + 4x^2)}{2} - \frac{3x^3}{2} \right] \, dx
\]

\[
= \int_0^2 \left[ 18x - 6x^2 - \frac{3x^3}{2} \right] \, dx
\]

\[
= \left[ 9x^2 - 2x^3 - \frac{3x^4}{8} \right]_0^2
\]

\[
= 36 - 16 - 6 = 14.
\]

The \( x \)-component of the center of mass is

\[
\bar{x} = \frac{M_y}{m} = \frac{14}{20} = \frac{7}{10}.
\]

Recall that the momenta of inertia, or the second moment, of a particle of mass \( m \) about an axis is \( mr^2 \) where \( r \) is the distance of the particle from the axis.

The **moment of inertia** of a lamina occupying a region \( D \) with density \( \rho(x, y) \) **about the** \( x \)-axis is

\[
I_x = \iint_D y^2 \rho(x, y) \, dA,
\]

the momenta of inertia **about the** \( y \)-axis is

\[
I_y = \iint_D x^2 \rho(x, y) \, dA,
\]

and the momenta of inertia **about the origin** is

\[
I_0 = \iint_D (x^2 + y^2) \rho(x, y) \, dA.
\]

Notice that \( I_0 = I_x + I_y \).

Example (continued). For the lamina in the previous examples,

\[
I_x = \frac{1104}{5}, \quad I_y = \frac{72}{5}, \quad I_0 = \frac{1176}{5}.
\]