Regularity Properties of Radon Measures

Definition (2.16). If $\mu$ is a positive measure on $\mathcal{M}$ in $X$, then an $E \in \mathcal{M}$ is $\sigma$-finite if $E = \bigcup_{i=1}^{\infty} E_i$ for $E_i \in \mathcal{M}$ with $\mu(E_i) < \infty$.

Proposition (Folland p. 209). If $\mu$ is a Radon measure, then $\mu$ is inner regular on all $\sigma$-finite Borel sets.

Proof: Suppose $E \in \mathcal{B}_X$ has finite measure. (Such an $E$ is trivially $\sigma$-finite.)

Let $\epsilon > 0$. By outer regularity, there is an open $U \supset E$ such that $\mu(U) < \mu(E) + \epsilon/2$.

Since $U = (U \setminus E) \cup (U \cap E) = (U \setminus E) \cup E$, then $\mu(U) = \mu(U \setminus E) + \mu(E)$, so that

$$\mu(E) + \epsilon/2 > \mu(U) = \mu(U \setminus E) + \mu(E)$$

from which $\mu(U \setminus E) < \epsilon/2$.

By outer regularity, there is an open $V \supset U \setminus E$ such that $\mu(V) < \epsilon/2$.

By inner regularity on open sets, there is a compact $F \subset U$ such that $\mu(F) > \mu(U) - \epsilon/2$.

Let $K = F \setminus V$.

Then $K$ is a compact subset of $E$.

[F compact $\implies$ $F$ closed because $X$ is Hausdorff (see Corollary (a) of Theorem 2.5); $V$ open; so $K = F \setminus V$ is a closed subset of a compact $F$ $\implies$ $K$ compact (see Theorem 2.4).]

Since $F = K \cup (F \cap V)$, then $\mu(F) = \mu(K) + \mu(F \cap V)$ with $\mu(F) < \infty$ and $\mu(K) < \infty$, so that

$$\mu(K) = \mu(F) - \mu(F \cap V)$$

$$> \mu(U) - \epsilon/2 - \mu(F \cap V)$$

$$> \mu(U) - \epsilon/2 - \epsilon/2$$

$$= \mu(U) - \epsilon$$

$$\geq \mu(E) - \epsilon.$$  

Thus, $\mu$ is inner regular on $E$.

Suppose $\mu(E) = \infty$ and $E$ is $\sigma$-finite.

Then $E = \bigcup_{i=1}^{\infty} E_i$ for $E_i \in \mathcal{B}_X$ with $\mu(E_i) < \infty$.

Set $H_j = \bigcup_{i=1}^{j} E_i$.

Then $E$ is an “increasing” union of $H_j$ with $\mu(H_j) < \infty$ and $\mu(H_j) \to \infty$ (recall Theorem 1.19(d)).

Thus for any $N \in \mathbb{N}$, there is a $j$ such that $\mu(H_j) > N$. 
This implies by the preceding argument (the \( \mu(E) < \infty \) case), that there is a compact \( K \subset H_j \) with \( \mu(K) > N \).

Hence \( \mu \) is inner regular on \( E \). □

**Definition** (Folland p. 24). A positive measure \( \mu \) on \( \mathcal{M} \) in \( X \) is \( \sigma \)-finite if \( X = \bigcup_{i=1}^{\infty} E_i \) with \( E_i \in \mathcal{M} \) and \( \mu(E_i) < \infty \); \( \mu \) is finite if \( \mu(X) < \infty \).

**Remark.** Finiteness of \( \mu \) implies \( \sigma \)-finiteness of \( \mu \).

**Corollary A** (Folland p. 209). If a Radon measure \( \mu \) is \( \sigma \)-finite, then it is regular.

**Proof:** Every \( E \in \mathcal{B}_X \) is \( \sigma \)-finite: \( E = \bigcup_{i=1}^{\infty} (E \cap E_i) \) with \( \mu(E \cap E_i) \leq \mu(E_i) < \infty \).

The Proposition on p.1 implies the inner regularity of \( \mu \) □

**Definition** (2.16). A set \( E \subset X \) is \( \sigma \)-compact if \( E = \bigcup_{i=1}^{\infty} K_i \) with \( K_i \) compact.

**Corollary B** (Folland p. 209). If \( X \) is \( \sigma \)-compact, then every Radon measure is regular.

**Proof:** \( \sigma \)-compactness of \( X \) implies \( \sigma \)-finiteness for Radon measure \( \mu \): \( E = \bigcup_{i=1}^{\infty} (E \cap K_i) \) where \( \mu(E \cap K_i) \leq \mu(K_i) < \infty \). □

**Positive Linear Functionals**

**Definition** (2.1). A linear functional on a complex vector space \( V \) is a map \( \Lambda : V \to \mathbb{C} \) such that

\[
\Lambda(\alpha f + \beta g) = \alpha \Lambda(f) + \beta \Lambda(g)
\]

for all \( \alpha, \beta \in \mathbb{C} \), and all \( f, g \in V \).

**Definition** (2.2). A linear functional \( \Lambda \) on \( C_c(X) \) is positive if \( \Lambda(f) \geq 0 \) whenever \( f(X) \subset [0, \infty) \).

**Examples.** Let \( \mu \) be a Borel measure for which \( \mu(K) < \infty \) for every compact \( K \subset X \) (for instance, \( \mu \) could be a Radon measure).

a) The map \( \Lambda : C_c(X) \to \mathbb{C} \) defined by

\[
\Lambda(g) = \int_X g d\mu
\]

is a positive linear functional on \( C_c(X) \).

Here \( \text{supp}(g) \) is compact, \( g \) is bounded on \( \text{supp}(g) \), and \( \mu(\text{supp}(g)) < \infty \); so the integral is finite.

Incidentally, this shows that \( \overline{C_c(X)} \subset L^1(\mu) \)

b) For a fixed positive \( f \in L^1(\mu) \), the map \( \Lambda : C_c(X) \to \mathbb{C} \) defined by

\[
\Lambda(g) = \int_X g d\varphi \text{ where } d\varphi = f d\mu
\]

is a positive linear functional on \( C_c(X) \).

Here \( \mu(K) < \infty \) implies \( \varphi(K) < \infty \) because \( \int_K d\varphi = \int_K f d\mu = \int_X \chi_K f d\mu \leq \int_X f d\mu < \infty \).
c) The **Dirac functional** $\Lambda : C_c(X) \to \mathbb{C}$ defined by $\Lambda f = f(x_0)$ for fixed $x_0$ is a positive linear functional.

The Riesz Representation Theorem asserts that every positive linear functional on $C_c(X)$ arises as integration against a Borel measure.

Positivity of a linear functional implies a strong form of continuity.

**Definition.** The **uniform norm** of $f \in C_c(X)$ is $\|f\|_u = \sup\{|f(x)| : x \in \text{supp}(f)\}$.

**Remark.** The uniform norm is a metric on $C_c(X)$.

**Proposition** (Folland p. 204). If $\Lambda$ is a positive linear functional on $C_c(X)$, then for each compact $K \subset X$ there is a constant $C_K$ such that $|\Lambda(f)| \leq C_K\|f\|_u$ for all $f \in C_c(X)$ with $\text{supp}(f) \subset K$.

**Proof.** It suffices to show this for a real-valued $f$.

For a compact $K$, choose $\varphi \in C_c(X)$ with $\varphi(X) \subset [0,1]$ and $\varphi = 1$ on $K$ (such exists by Urysohn’s Lemma).

The if $\text{supp}(f) \subset K$, we have $|f| \leq \|f\|_u \varphi$, that is, $\|f\|_u \varphi - f \geq 0$ and $\|f\|_u \varphi + f \geq 0$.

Thus $\|f\|_u \Lambda(\varphi) - \Lambda(f) \geq 0$ and $\|f\|_u \Lambda(\varphi) + \Lambda(f) \geq 0$, so that $|\Lambda(f)| \leq \Lambda(\varphi)\|f\|_u$. □

The Riesz Representation Theorem for Positive Linear Functionals on $C_c(X)$ (2.14). If $X$ is a LCH space, and $\Lambda$ is a positive linear functional on $C_c(X)$, then there is a $\sigma$-algebra $\mathcal{M}$ in $X$ for which $\mathcal{B}_X \subset \mathcal{M}$, and a unique positive measure $\mu$ on $\mathcal{M}$ such that

a) $\Lambda f = \int_X f\,d\mu$ for all $f \in C_c(X)$,
b) $\mu(K) < \infty$ for every compact $K$ in $X$,
c) $\mu(E) = \inf\{\mu(V) : V \supset E, V \text{ open}\}$ holds for every $E \in \mathcal{M}$, ("outer regularity"),
d) $\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ compact}\}$ holds for every open $E \in \mathcal{M}$, and for every $E \in \mathcal{M}$ with $\mu(E) < \infty$, ("partial inner regularity"),
e) $E \in \mathcal{M}$ with $\mu(E) = 0$ implies $A \in \mathcal{M}$ for all $A \subset E$ ($\mu$ on $\mathcal{M}$ is complete).

In particular, the restriction of $\mu$ to $\mathcal{B}_X$ is a Radon measure.

**Remark.** If $X$ is $\sigma$-compact, then the Radon measure $\mu|\mathcal{B}_X$ is regular by Corollary B, and the positive measure $\mu$ on $\mathcal{M}$ is the completion of $\mu|\mathcal{B}_X$ by Theorems 1.36 and 2.17(b) (part (b) in these notes, part (c) in Rudin).

**Proof.** Throughout the proof, $K$ stands for a compact subset of $X$ and $V$ for an open subset of $X$.

Uniqueness. If $\mu$ satisfies c) and d), then $\mu$ is determined by its values on compact sets: by d), $\mu$ is determined for every open subset of $X$ by compact subsets $K$, and by c), $\mu$ is determined for every $E \in \mathcal{M}$ by open subsets.

It thus suffices to show that $\mu_1(K) = \mu_2(K)$ for all $K$ whenever $\mu_1, \mu_2$ are two measures for which the theorem holds.

Fix $K$ and $\epsilon > 0$. 
By b) and c) there is a $V$ such that $K \subset V$ and $\mu_2(V) < \mu_2(K) + \epsilon$.

[Part b) gives the finiteness of $\mu_2(K)$ so that $<$ makes sense in $\mu_2(V) < \mu_2(K) + \epsilon$.]

By Urysohn’s Lemma, there is an $f \in C_c(X)$ with $K \prec f \prec V$, i.e., $f = 1$ on $K$, supp$(f) \subset V$.

Hence

$$
\mu_1(K) = \int_X \chi_K d\mu_1
\leq \int_X f d\mu_1 \quad [K \prec f]
= \Lambda f \quad \text{[Part (a) for $\mu_1$]}
= \int_X f d\mu_2 \quad \text{[Part (a) for $\mu_2$]}
\leq \int_X \chi_V d\mu_2 \quad [f \prec V]
= \mu_2(V)
< \mu_2(K) + \epsilon.
$$

Thus $\mu_1(K) \leq \mu_2(K)$.

Switching the roles of $\mu_1$ and $\mu_2$ gives the opposite inequality, and so $\mu_1(K) = \mu_2(K)$.

Since $K$ was arbitrary, we have $\mu_1 = \mu_2$ on $\mathcal{M}$.