An equivalence for the Riemann Hypothesis in terms of orthogonal polynomials

June 20-25, 2004

Canadian Number Theory Association VIII

David A. Cardon (BYU)
Outline:

1. Brief review of orthogonal polynomials.

2. Orthogonal polynomials related to \( \zeta(s) \).

3. Equivalence for RH with simple zeros in terms of orthogonal polynomials.

4. Remarks on proof.

Main Ideas: Approximate the Riemann \( \zeta \)-function with orthogonal polynomials.
Definition: A distribution function is a non-decreasing real-valued function such that its moments exist for all \( n = 0, 1, 2, \ldots \).

\[
\left( x \right) \phi p(x) b(x) d = \int_{\infty}^{\infty} \left( (x) b, (x) d \right)
\]

Inner Product: The spectrum of \( \phi \) is the set \( \{ 0 < \varrho \} \) for all \( \varrho > 0 \) for all \( x = 0, 1, 2, \ldots \).

\[
\{ 0 < \varrho \} \left( \phi \right) \left( \varrho \right) = \int_{\infty}^{\infty} \left( (x) b, (x) d \right) = 0
\]

Definition: A distribution function is a non-decreasing real-valued function such that its moments exist for all \( n = 0, 1, 2, \ldots \).
Real Orthogonal Polynomials

When the spectrum $S$ is infinite, orthogonalize

$$\{ (x)^0 \}$$

using Gram-Schmidt

$$\sum_{k=0}^{\infty} x^k$$

When the spectrum $S$ is finite, orthogonalize

Real Orthogonal Polynomials
\[
\mathcal{H}_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}
\]

**Example - Hermite Polynomials:**

\[
(x)_u \eta u \mathcal{P} = (x)^u d
\]

**Orthogonal Polynomials:**

\[
x_p \mathcal{P} x^{-\mathcal{P}}(x) b(x) d \infty \int \frac{\psi}{1} = \langle (x)b, (x)d \rangle
\]

**Inner Product:**

\[
\mathcal{P} \mathcal{P} = (\phi) S
\]

**Spectrum:**

\[
\mathcal{H} \mathcal{P} x^{-\mathcal{P}} \int \frac{\psi}{1} = (x) \phi
\]

**Distribution Function:**
Density Function:

\[ (y - x) \sum_{|y| = -\infty}^{\infty} = (x) p \]

Discrete Spectrum:

\[ Z = (\phi)^{S} \]

Distribution Function:

\[ (y - x) \sum_{|y| = -\infty}^{\infty} = (x) p \]

Unit Step Function:

\[
\begin{cases}
0 < x & 1 \\
0 > x & 0
\end{cases} = (\mathcal{U}) n
\]

Example - Step Function
The Riemann hypothesis is true if and only if $\xi\left(\frac{z}{2}\right) = (z)\Xi$.

\[
\left(\frac{1}{2} + \frac{2}{\pi} \right)^s = \left(\frac{z}{2}\right)\Xi
\]

\[
\left(s\right)^\frac{1}{2}\frac{1}{2} \left(s^{-\frac{1}{2}} - \frac{1}{2} - s\frac{1}{2}\right) = (s)\xi
\]

\[
s_{u} = \left[s_{u}\right]_{\infty} = (s)\xi
\]

Riemann $\xi$-function
\[ F(T) = \frac{1}{2} \int_{-\infty}^{\infty} f(z) dz \]

Distribution function related to \((\mathcal{L})_P\)
corresponds to $\frac{\pi}{2}$ and $\frac{\pi}{4}$ are positively oriented boundary of rectangular region with

\[ \forall \varepsilon > 0, \quad |(\hat{n} + x)f| > 0 < (z)f \text{ is real for real } z \]

Let $(z)f$ be analytic in $\mathbb{R}$ and satisfy

\[ \{ \pi/2 \leq \hat{n} \leq \pi/4, \quad 0 \leq x : \mathbb{C} \ni \hat{n} + x = z \} = \mathbb{R} \]

\[ \int_{\Gamma} \frac{\lambda z}{\lambda^2} = (\mathcal{L}) f \]

$s$-Distribution function related to $(\mathcal{L}) f$
Orthogonal polynomials related to zeros of $\Re(z)$ by $0 < \Re(z)$ in region $(s)$. Let $f_p(z)$ be the orthogonal family relative to measure $d\mu$. Define $F(T)$ for $T > 0$.\[ F(T) = \int_{\gamma<T} f(z) \, d\mu \]

with $\gamma < \infty + 1$.

The zeros label of $\Re(\xi)$ in the region is counting multiplicity. Orthogonal polynomials related to zeros of $\Re(z)$.\[ \{(\gamma_i \xi - \gamma_n \xi) f + (\gamma_i \xi + \gamma_n \xi) f\} \bigcap_{0<\gamma_i \xi} + (\gamma_n \xi) f \bigcap_{0=\gamma_i \xi} = (\gamma) F \]

If $\Re(\xi) \neq T$, then $w_i < \infty$.
The Main Theorem

Theorem: The Riemann Hypothesis with simple zeros is true if and only if

\[
\lim_{n \to 1} \frac{(0)^{1+u \zeta d} z}{(z)^{1+u \zeta d}} \left[ \prod_{i=1}^{\infty} \left( 1+\frac{u \zeta d}{z} \right) \right] = \lim_{n \to 1} \frac{(0)^{u \zeta d}}{(z)^{u \zeta d}} \left[ \prod_{i=1}^{\infty} \left( 1+\frac{u \zeta d}{z} \right) \right]
\]

Note: The proof also shows that

\[
\lim_{n \to 1} \frac{(z/2)^{1+u \zeta d} z}{(z/2)^{1+u \zeta d}} = \lim_{n \to 1} \frac{(0)^{u \zeta d}}{(z)^{u \zeta d}} \left[ \prod_{i=1}^{\infty} \left( 1+\frac{u \zeta d}{z} \right) \right]
\]

true if and only if

The Main Theorem
1. Remarks about the proof

   1. Uses classical facts about real orthogonal polynomials. For example,

   (a) Zeros of \( p_n(z) \) are real and simple
   (b) Zeros of \( p_n(z) \) and \( p_{n+1}(z) \) interlace
   (c) Zeros of \( \{z\}^{\nu d} \) are related to \( S \)

2. The \( n \)th moments

\[
\mathcal{H}P(x) b(x) \int_{-\infty}^{\infty} = \langle (x) b, (x) d \rangle
\]

3. Inner product: Use fact that \( \mathcal{N}(T) \sim T \frac{\nu}{\lambda} \),

   Proof: Exist

\[
\mathcal{H}P u x \int_{-\infty}^{\infty} = u f
\]
Theorem.

This is one of the most important steps of the proof of the

\[
\int_0^\infty = \sum_{n=1}^{\infty} \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{x^n}{n!}
\]

to show that

Proof: Use Carleman's Criterion. Make careful estimates

are "substantially equal".

\[
\cdots = u \int_0^\infty x^0 = u \int_0^\infty \phi P_u x = u \phi
\]

solve the moment problem

is determined. Thus, all distribution functions (x(h) that

\[
\cdots = u \int_0^\infty \phi P_u x = u \phi
\]

moments of the distribution function

Show the "Hamblenur moment problem"
Thespectrum of $F$ consists of all $\gamma$ such that $\gamma + i\gamma \theta$ is a zero of $\zeta(1/2 + iz)$. Label positive values of $\gamma$ (without multiplicity) as $a_1 < a_2 < a_3 < \cdots$ 

Show that $a_k = \lim_{n \to \infty} x_{2n^{k-1}}$.

Label positive zeros of $p_n(x)$ as $x_{2n^2} > \cdots > x_{2n} > x_{2n^1}$.
6. The sequence of polynomials

\[ p_n(z) \]

converges uniformly on compact sets to the entire function with simple real zeros corresponding to the real parts of the zeros of

\[ \frac{z}{2} + \frac{1}{2} i \cdot \zeta \]

RH with simple zeros true,

\[ \lim_{n \to \infty} \frac{p_n(z)}{z} = \prod_{1=0}^{\infty} \frac{(0)^{u \zeta d} \infty \leftarrow u}{(z)^{u \zeta d} \infty \leftarrow u} \]

\[ \Leftrightarrow \text{RH with simple zeros true} \]

7. RH with simple zeros true

\[ \lim_{n \to \infty} \frac{p_n(z)}{z} = \prod_{1=0}^{\infty} \frac{(0)^{u \zeta d} \infty \leftarrow u}{(z)^{u \zeta d} \infty \leftarrow u} \]

zeros of \( \frac{z}{2} + \frac{1}{2} i \cdot \zeta \).

Converges uniformly on compact sets to the entire function

\[ \frac{(0)^{u \zeta d}}{(z)^{u \zeta d}} \]