THE COMBINATORIAL STRUCTURE OF THE HAWAIIAN EARRING GROUP
March 30, 1999

J.W. CANNON, G.R. CONNER

Abstract. In this paper we study the combinatorial structure of the Hawaiian earring group, by showing that it can be represented as a group of *transfinite words* on a countably infinite alphabet exactly analogously to the representation of a finite rank free group as finite words on a finite alphabet. We define a *big free group* similarly as the group of transfinite words on given set, and study their group theoretic structure.

[1]

1. Introduction

The Hawaiian earring is the metric wedge of a countable null sequence of circles. Since it is not semilocally simply connected, covering space theory is not adequate for the analysis of its fundamental group. Its group is uncountable and is not free (though, in some sense, it is almost free).

Figure 1. Hawaiian Earring

The motivating goal of our work is supplied by the following question: suppose that G is the fundamental group of a space which is not locally

[1]This research was supported in part by an NSF research grant.

simple, so that covering space theory is inadequate for its analysis; what is the cleanest combinatorial description one can give for G?

1.1. **Seeking a description by means of transfinite words.** Our goal is to give a combinatorial description of the Hawaiian earring group which has two principal features:

(1) The combinatorial description stays very close to the two fundamental features of the Hawaiian earring itself: one–dimensionality and countability. (In contrast, any topological description of the group via CW–complexes would either require uncountably many cells of both dimensions 1 and 2 or would involve infinitely many CW–complexes and connecting homomorphisms. In the equivalent terminology of standard combinatorial group theory, the group requires either uncountably many generators and relators, or it requires description as an inverse limit of countable groups.)

(2) The combinatorial description maintains the spare, familiar features of free groups, namely, representation of elements by words, multiplication by concatenation, reduction by cancellation, with unique reduced representatives, free isometric action on a (nonstandard) tree, etc. In fact, the theory is a generalization of the theory of free groups and the insights gained in generalization cast interesting light on the classical theory.

As an introduction to the theory, we now describe the infinite words on which the theory is based. The basic object in the new theory replacing the notion of "word" in free groups is the *transfinite word*. A transfinite word (from now on we will just call it a "word") will be a map from a totally ordered set into a set of letters (the *alphabet*) and their inverses satisfying the property that the preimage of each letter is finite. Multiplication of two such words is defined by the obvious ordered concatenation of domains. The identity is the word with empty domain. The *reverse* (or *formal inverse*) of a word, the word obtained by reversing the order on the domain and inverting the labels, represents the inverse group element. We asked W.B.R. Lickorish what we might call these *big words* and he suggested that we call them *sesquipedalian*, an adjective used to describe people who use unnecessarily long words in conversation.

Now, at first glance this new definition of "word" might be a bit mysterious, and seem to have very little to do with standard notion of word. However it is very natural if one considers the problem of defining (possibly infinite) products of infinite words. For instance what should be the product of the word $x_1x_2x_3\dots$ followed by the word $\dots y_{-1}y_0y_1\dots$? If one allows ordered sets for the domains, one can allow the natural product

$$x_1 x_2 x_3 \dots \quad \dots y_{-1} y_0 y_1 \dots .$$

To fix ideas, we note that the inverse of the above word is the word

$$\dots {y_1}^{-1} {y_0}^{-1} {y_{-1}}^{-1} \dots \quad \dots {x_3}^{-1} {x_2}^{-1} {x_1}^{-1}.$$

The finiteness property which requires that no letter be used infinitely often in a single word is necessary to rule out absurdities of the form

$$x^\infty \cdot x = x^\infty \Rightarrow x = 1.$$

If we restrict ourselves to a finite alphabet our group turns out to be a finite rank free group, but if we allow a countably infinite alphabet we obtain the Hawaiian earring group.

As in the case of finite alphabets, different words may represent the same group element. The appropriate notion of cancellation is more subtle than one might at first presume. For now, it suffices to consider a single example.

$$x_1 {x_1}^{-1}$$

$$x_1 (x_2 {x_2}^{-1})(x_3 {x_3}^{-1}) {x_1}^{-1}$$

$$x_1 (x_2 (x_4 {x_4}^{-1})(x_5 {x_5}^{-1}) {x_2}^{-1})(x_3 (x_6 {x_6}^{-1})(x_7 {x_7}^{-1}) {x_3}^{-1}) {x_1}^{-1}$$

$$\vdots$$

The sequence is defined by at each step inserting two inverse pairs between any two adjacent inverse letters in the previous step. One sees that this word should represent the identity, but that in the limit no two inverse letters will be consecutive – indeed, no two words which are reverses of one another are consecutive.

1.2. **Technical Attractions.** The treatment has the following additional technical attractions:

(3) The method of study can be described as that of considering infinite words as topological objects. The proofs illustrate a large variety of standard topological techniques.

For example, representative words and cancellations arise naturally via simple applications of *transversality* or *general position.* Reduced representatives are analyzed via the *Jordan curve theorem.* As in the free group case, reduced representatives exist and are unique. However where uniqueness is a matter of finite induction in the free group case (all that is required there is that one cancel xx^{-1} pairs, and the only nonuniqueness is in which way one cancels $xx^{-1}x$ – to the left or to the right) the general case requires the full power of axiom of choice and careful application of the separation properties of the plane. In fact, the proof in the general case is, we feel, more informative when applied to free groups than the standard proof.

The group is embedded in an inverse limit of free groups via an application of the *Tychonoff product theorem for compact spaces.* The interesting point here is that the space which is proved to be compact is the space of all possible cancellations of a single (transfinite) word.

The commutator subgroup is analyzed via a study of *nonparallel curves in a surface* and *Euler characteristic.* Each word in the commutator subgroup appears as a "*foliation*" of a punctured surface.

After a bit of introspection it is clear that the initial/terminal subword pairs of an infinite word are in one–to–one correspondence with the *Dedekind cuts* of the domain.

Large classically-free subgroups are constructed via *Dedekind cuts*, *linearly-ordered topological spaces*, *compact, countable Hausdorff spaces*, *derived sets of limit points*, and *transfinite induction.*

Another highlight is the following:

(4) The combinatorial structure of the Hawaiian earring group suggests an obvious new family of groups, the big free groups, not free in the classical sense, but freer than free in the sense of allowing transfinite multiplication, transfinite cancellation rules, having unique transfinite reduced word representatives. These big free groups are of arbitrary cardinality and share many properties with the classically free groups and the Hawaiian earring group. In fact, they are the groups obtained by allowing alphabets of arbitrary cardinality. They contain classically free subgroups of huge cardinality yet have only limited homomorphisms into free Abelian groups.

1.3. **Related Papers.**

1.3.1. *History.* The first reference to the Hawaiian earring group of which we are aware is in G. Higman's paper, [H]. In this paper the author studies *unrestricted free products* of groups. He describes the unrestricted free product of countably many copies of $\mathbb{Z}$ and proves that this group is not a free group and that each of its free quotients has finite rank. In the last section of this paper he defines a subgroup P of this product and proves that P itself is not free and that any freely irreducible subgroup of P is either $\mathbb{Z}$ or the trivial group. It turns out that, although it is not proven in [H], P is the Hawaiian earring group. In [G], H. B. defines a subgroup of the unrestricted free product of groups which he calls the "topological free product". In the case where the factors consist of a countable collection of copies of $\mathbb{Z}$, the topological free product corresponds to P. He also proposes to show that P is the fundamental group of the Hawaiian Earring by embedding P into an inverse limit of free groups. In a later paper, J.W. Morgan and I. Morrison, [MM], point out that the proof in [G] that the proposed embedding is injective is in error; they give a correct proof. Finally in [dS], B. deSmit gives straightforward proofs that the Hawaiian earring group embeds in an inverse limit of free groups, gives a characterization of the elements of the image and uses this to prove that the set of homomorphisms from the Hawaiian earring group to $\mathbb{Z}$ is countable, and that the Hawaiian earring group is uncountable, and that consequently the Hawaiian earring group is not free.

1.3.2. *Recent Results.* There are several other authors who are currently working in related areas. We now give a short synopsis of those preprints of which we are aware.

In [E2], Eda independently builds an algebraic theory of "infinitary words" similar to our own transfinite words. Our $BF(c)$ corresponds to his *complete free product* $\times_c\mathbb{Z}$. In particular,

$$\pi_1(\mathbb{H}_I, o) \simeq \times_I^\sigma \mathbb{Z} \text{ and } \Pi(\mathbb{H}_I, o) \simeq \times_I \mathbb{Z}.$$

Eda introduced the notions which correspond to the notions of reduced word and the strong abelianization given in the present paper. Eda gives algebraic proofs to a number of results which are similar to results proved using topological means in Section 3 and Section 4. His approach to this topic is algebraic and concerns itself with inverse limits. Our approach differs in that our primary goals are to understand the combinatorial and topological structures of infinite words as independent objects. In [E2], Eda also investigates the notion of a *noncommutatively slender* group. The relationship between this notion and our work is discussed in Section 7. In [Z1] A. Zastrow argues that any subset of $\mathbb{R}^2$ has trivial higher homotopy. In [Z2] Zastrow shows that the so called "non-abelian Specker group" (in our terminology this is the subgroup of the Hawaiian earring group consisting of those elements which can be represented by transfinite words in which for each word there is a uniform bound on how many times each letter appears) is a free subgroup of the Hawaiian earring group. This is related to Section 5 in the current paper. In [Z3] Zastrow shows that a certain (not finitely generated) subgroup of the Hawaiian earring group has a free action on an $\mathbb{R}$-tree but is not a free product of surface groups and free abelian groups. This is related to Section 6 in the current paper. In [S], Sieradski defines the notion of *order–type words* on *weighted groups*. He calls the weighted group, G along with a *word evaluation* an *Omega–group.* Briefly, an order type word is a map from the components of the complement of a closed nowhere dense subset of $[0, 1]$ containing both 0 and 1, to the group G with the property that short intervals are mapped to group elements of small weight. A word evaluation is a map from the set of order–type words to the group G satisfying several technical axioms. A special case is that of *free Omega–groups*. In that situation the order–type words map into an alphabet X rather than into the group itself and the technical axioms are vacuous. Free omega–groups are related to big free groups of in the sense that both are groups of transfinite "words" and each construction was designed, in part, to put a combinatorial structure on the Hawaiian earring group and to give generalizations of its structure. In [BS1], W.A. Bogley and Sieradski develop a homotopy theory suitable for studying their *weighted combinatorial group theory*. In [BS2] they study a notion of *universal path space* related to the big Cayley graphs appearing in Section 6 in the current article. They give a metric (which is not a path metric) in which the big Cayley graph is contractible and one–dimensional (though not a simplicial tree), in such a way that the underlying action of the group is isometric.

1.3.3. *The Current Project.* This paper is the first in a series of three papers. Our treatment of the topics in this first paper suggests some obvious questions:

(5) D. Wright has asked whether the big free groups of higher cardinality might arise as fundamental groups of *generalized Hawaiian earrings* of some sort. After giving him the obvious answer that loops are simply not long enough to traverse uncountably many circles, we reconsidered our answer and discovered a new fundamental group whose loops are based not on the real unit interval as parameter space but on arbitrary compact, connected, linearly-ordered intervals. Generalized Hawaiian earrings, this new *big fundamental group*, and the big free groups will be united in a second paper, [CC2], of this series where it is shown that the big fundamental group of a generalized Hawaiian earring is a big free group.

(6) The Hawaiian earring is only the simplest of the locally complicated spaces. How does its fundamental group relate to that of the Sierpinski curve or the one–dimensional Menger curve? We broach that subject in a third paper of the series, [CC3]. We show that much of the theory generalizes: group elements are represented by a new kind of word, there is a cancellation theory for words related intimately to R. L. Moore's famous theorem, [RLM], on cell-like upper semicontinuous decompositions of the plane, there is a notion of reduced words, and elements have unique reduced representatives, the groups embed in an

inverse limit of finitely generated free groups, etc. We study the relationship between the countability of fundamental groups and the existence of a universal covering space. We study homomorphisms of these groups into countable or Abelian groups. We prove that the fundamental group of a connected, locally path connected one–dimensional, second countable metric space is countable if and only if it is free if and only if the space has a universal cover.

The Hawaiian earring and its fundamental group have long been favorites of group theorists and topologists. Thus a number of our results either appear in the literature or are folklore. We shall attempt to point these out as occasion arises. We believe, however, that our combinatorial description, its generalization to arbitrary cardinality, and our proofs form an attractive package that is clean enough and enlightening enough to warrant attention.

1.4. **Outline.** The outline of this first paper in a series of three is as follows:

In Section 2 we analyze the Hawaiian earring group via transversality. We thereby extract the essential combinatorial features which will be used to define and study the big free groups of arbitrary cardinality in Section 3. We recall and prove the famous properties of the Hawaiian earring and its fundamental group which have made the earring important as a topological example. Portions of the proofs are postponed to their more natural setting among all of the big free groups.

In Section 3 we define the big free groups and establish their fundamental properties. A big free group is not just a group; it has additional structure which allows rather arbitrary transfinite multiplication. We observe that the Hawaiian earring group admits this extra structure, and, as such, is the big free group of countable cardinality.

In Section 4 we study the commutator subgroup and the abelianization of the Hawaiian Earring group and its generalizations, the big free groups. We also define the notion of the *strong abelianization* of the Hawaiian Earring group and give an algebraic description of this geometrically defined object. Finally we study homomorphisms from the Hawaiian Earring group into countable and Abelian groups. We use the knowledge obtained from this study to give a second proof of the fact that the Hawaiian Earring group is not free by showing that any map from the Hawaiian Earring group into a free Abelian group has finitely generated image.

In Section 5 we undertake study the topological structure of individual transfinite words. We define the *space of a reduced transfinite word* to be the space of Dedekind Cuts in its domain. We show the set of reduced words whose spaces are countable form a free group of uncountable rank in any big free group whose alphabet it infinite (including the Hawaiian Earring group).

In Section 6 we introduce the *big Cayley graphs* of the big free groups. These graphs are generalized trees on which the big free group acts freely and "isometrically" with respect to a certain generalized non-Archimedean metric.

Finally, in Section 7 the referee discusses the relationship between the results in this paper and the notion of noncommutatively slender groups investigated by Eda in [E2].

2. The Hawaiian earring

The Hawaiian earring E is the compact subset of the xy-plane that is the union of the countably many circles C_n, where C_n has radius $1/n$ and center $(0, -1/n)$. The earring is obviously both locally and globally path connected. We choose the origin $\mathbf{0} = (0, 0)$ as base point and define G to be the fundamental group $\pi_1(E, \mathbf{0})$. There are countably many elements of G which we wish to distinguish by the name of *basis*, namely those elements a_n represented by the loop which traverses the circle C_n exactly once in the counterclockwise direction.

The earring E so clearly resembles the CW–complex which is the wedge of countably many circles, that one is tempted to conjecture that its fundamental group is a free group with free basis given by the elements a_n. This conjecture is false, for it would imply that the group G is a countable group, which Theorem 2.5 contradicts. We shall see however in Section 3 that the elements a_n form a basis for G with respect to a transfinite law of multiplication.

In this section we first study the combinatorial properties of the Hawaiian earring group. We then confirm its most famous classical properties, namely that it is uncountable and not free. We finish this section with a result about the cone over the Hawaiian earring. This result has been in the folklore for at least thirty years; a proof appears in [Z4]. We include these three well-known results to show how nicely they mesh with our topological point of view.

2.1. The combinatorial properties of the Hawaiian earring group.

The structure of the Hawaiian earring, as a subset of the plane, disguises its very homogeneous nature: any permutation of the circles can be realized by a homeomorphism of the earring. Can one then describe the earring in a more homogeneous way that does not depend upon the size of the circles? We note that the complement of the base point is the disjoint union of countably many open arcs. We then examine a neighborhood of the base point: it includes all but finitely many of the open arcs and omits only a compact set of the remaining. That is, the Hawaiian earring is precisely the one-point compactification of a countable disjoint union of open arcs.

Theorem 2.1. *Consider an arbitrary function $f : X \longrightarrow E$. For each of the open intervals I_α from which E was formed, define X_α to be the inverse of I_α under f and define Y_α to be the closure of X_α in X. Then the function f is continuous if and only if each of the sets X_α is open and each of the functions $f_\alpha = f|Y_\alpha$ is continuous.*

Proof. In order that f be continuous it is clear that the stated conditions concerning X_α and f_α must be satisfied. Suppose conversely that they are satisfied. We shall then prove that f is continuous at each point x of X. To that end we fix a neighborhood N of $f(x)$ in E.

Case 1. Suppose that $x \in X_\alpha$. Since f_α is continuous on Y_α, there is an open set U in X such that $f(U \cap Y_\alpha) \subset N$. Then $U \cap X_\alpha$ is a neighborhood of x in X that is mapped by f into N. Thus f is continuous at x.

Case 2. Suppose that $f(x)$ is the base point ∞ of E, the compactification point. Then there are compact sets $C_1, \ldots, C_k$ in finitely many $I_1, \ldots, I_k$ of the constituent open intervals such that $N = E \setminus (C_1 \cup \cdots \cup C_k)$. The set $f_j^{-1}(C_j)$ is closed in the closed set Y_j; hence its complement U_j is open in X and contains x. The intersection $U = U_1 \cap \cdots \cap U_k$ is open in X, contains x, and maps into the complement of $(C_1 \cup \cdots \cup C_k)$, that is, into N. We conclude that f is continuous at x. □

In each of the constituent circles C_n of E we let p_n be the bottommost point $(0, -2/n)$. We set $P = \{p_1, p_2, \ldots\}$. We note that $E \setminus \{\mathbf{0}\}$ is a 1-manifold, that P is a closed zero–dimensional submanifold, and that $E \setminus P$ is contractible (either by obvious construction or by Theorem 2.1).

We use $S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$ for the unit circle, $B^2 = \{(x, y) \mid x^2 + y^2 \leq 1\}$ for the unit disk, both with base point $\mathbf{1} = (1, 0)$.

Elements of G are represented by maps $f : S^1 \longrightarrow E$ such that $f(\mathbf{1}) = \mathbf{0}$. A map $f : S^1 \longrightarrow E$ represents the trivial element if and only if f extends to a map $F : B^2 \longrightarrow E$.

Maps f and F can always be adjusted homotopically so that they are transverse to the submanifold P. One need only perform the homotopies locally and individually within the individual circles C_n by Theorem 2.1. If $F|S^1$ is already transverse, then F may be adjusted without changing $F|S^1$.

We shall therefore always assume f and F transverse to P before any discussion begins. As a consequence, $f^{-1}(P) \subset S^1 \setminus \{\mathbf{1}\}$ will always be a countable set with each point $s \in f^{-1}(P)$ isolated and surrounded by a small closed interval $I(s)$ in S^1 which maps homeomorphically to a small interval $J(s)$ surrounding p_n in some $C_n \setminus \{\mathbf{0}\}$; we may assume all of the intervals $I(s)$ disjoint. For the map F, the set $F^{-1}(P)$ will consist of isolated arcs and simple closed curves, the arcs joining pairs of points of S^1, the simple closed curves lying in $\mathrm{int}(B^2)$. We can always remove the closed curves homotopically if we wish.

We now extract the essential combinatorial information supplied by the map f. We define $S = S_f = f^{-1}(P)$, and we totally order S counterclockwise in the open interval $S^1 \setminus \{\mathbf{1}\}$. We define A to be the alphabet $\{a_1, a_2, \dots\}$ and define a map $\alpha = \alpha_f : S \longrightarrow A \cup A^{-1}$ as follows: for each $s \in S$ we order $I(s)$ counterclockwise, we let $f(s) = p_n \in C_n \cap P$, and we define $\alpha(s)$ to be a_n or a_n^{-1} as the ordered image interval $J(s) = f(I(s))$ crosses p_n in the counterclockwise or clockwise direction. The ordered set S and the function $\alpha : S \longrightarrow A \cup A^{-1}$ represent f combinatorially.

With the map $F : B^2 \longrightarrow E$ having boundary map f we associate the pairing $* : S_f \longrightarrow S_f$ which associates to each $s \in S_f$ the point $s^* \in S_f$ to which it is joined by an arc of $F^{-1}(P)$. This pairing represents F combinatorially.

The key combinatorial features of α and $*$ are the following.

$\alpha 1$: S is totally ordered.
$\alpha 2$: $\alpha^{-1}(a)$ is finite for each $a \in A \cup A^{-1}$.
$1^\star$: $*$ is a fixed-point-free involution of the set S_f.
$2^\star$: $*$ is *noncrossing* in the sense that if $s_1, s_2 \in S_f$, and if $s_1 < s_2 < s_1^*$, then $s_1 < s_2^* < s_1^*$
$3^\star$: $*$ is an *inverse pairing* in the sense that $\alpha(s^*) = \alpha(s)^{-1}$ for all $s \in S_f$.

Here are the key geometric facts.

Theorem 2.2. *If α satisfies $\alpha 1$ and $\alpha 2$, then there exist an order-preserving embedding of S in $S^1 \setminus \{\mathbf{1}\}$ and a function $f : S^1 \longrightarrow E$ such that $\alpha : S \longrightarrow S$ represents f.*

Proof. Let $s_1, s_2, \dots$ be an enumeration of S. Embed s_1 arbitrarily in $S^1 \setminus \{\mathbf{1}\}$ and wrap a small interval neighborhood around the circle C_n corresponding to generator $\alpha(s_1)$. Inductively, embed s_k in the right order in S^1 and wrap a small closed interval neighborhood of s_k around the appropriate C_n in the correct direction indicated by $\alpha(s_k)$. Finally map the remainder of S^1 to $\mathbf{0} \in E$. Continuity is immediate from Theorem 2.1. □

Theorem 2.3. *If $\alpha : S \longrightarrow S$ is represented by $f : S^1 \longrightarrow E$, then f is nullhomotopic if and only if there is a pairing $* : S \longrightarrow S$ satisfying $1^\star$, $2^\star$, and $3^\star$.*

Proof. For each pair (s, s^*), join s and s^* in $S^1 \setminus \{\mathbf{1}\}$ by a circular segment (or diameter) $A(s, s^*)$ in B^2 orthogonal to $S^1 = \partial(B^2)$. Map $A(s, s^*)$ to $f(s) = f(s^*)$. The arcs $A(s, s^*)$ may accumulate on other circular segments A perpendicular to S^1. Map each such segment A to $\mathbf{0}$. Continuity to this point follows from Theorem 2.1 if we use the fact that only finitely many arcs have been mapped into any $C_n \setminus \{\mathbf{0}\}$. We have now extended f to a map on a closed subset X of B^2 such that each component U of $B^2 \setminus X$ is the interior of a disk D in B^2, and $f|\partial D$ does not cross any of the points p_n because of $3^\star$. Hence $f|\partial D$ is nullhomotopic in the set $f(\partial D)$. We choose such an extension of f to each D. The resulting map $F : B^2 \longrightarrow S^1$ is again continuous by Theorem 2.1 since, on any of the closed sets Y_n, the map F is a finite union of continuous functions on closed sets, namely of those involving the finitely many arcs mapped into C_n. □

Definition 2.4 (Infinitely Divisible)**.** An element h of a group H is said to be *infinitely divisible* if there are infinitely many integers e and corresponding nonidentity elements h_e of H so that $(h_e)^e = h$. The subgroup of H generated by infinitely divisible elements is normal. A group with no infinitely divisible elements is torsion–free.

The following result outlines many of the standard results on the Hawaiian earring group and gives simple proofs. Parts (1) and (2) were proven by Griffiths [G], (3) and (4) were proven by Higman [H] and (5) and (6) follow from work of Eda [E2].

Theorem 2.5. *Let G denote the Hawaiian earring group, and let G^{ab} be the abelianization of G, (and consequently, $H_1(H)$, the first homology of the Hawaiian earring.) Then*

1. *G is uncountable.*
2. *G embeds in an inverse limit of free groups.*
3. *G Every finitely generated subgroup is free.*
4. *G is not free.*
5. *G^{ab} has infinitely divisible elements.*
6. *G^{ab} cannot be embedded in an inverse limit of free abelian groups.*
7. *G^{ab} is not a free abelian group.*

Remark 2.5.1. We shall see in Section 4 that the group G contains uncountably generated free groups but that it cannot be mapped homomorphically onto even a countably generated free group or free Abelian group. Two other proofs that G is not free appear in Section 4.2 and Section 4.3.

Proof. We see that the Hawaiian earring group G is uncountable as follows. Let $i = (i_1, i_2, \dots)$ denote any binary sequence. Map $[0, 1]$ into E as follows. Let I_n denote the interval with initial point $1/2^n$ and terminal point $1/2^{n-1}$. If $i_n = 0$, then map the entire interval I_n to the base point. If $i_n = 1$, then wrap the interval I_n once around the circle C_n in the counterclockwise direction so as to represent the element a_n. Map $0 \in [0, 1]$ to the base point. Clearly the resulting map

f_i is continuous and represents an element of G. Given two distinct binary sequences i and j, choose an integer m such that $i_m \neq j_m$. Let $p_m : E \longrightarrow C_m$ denote the natural retraction map which takes all of the circles other than C_m to the base point. Then $p_m \circ f_i$ and $p_m \circ f_j$ are clearly not homotopic. Therefore, f_i and f_j define distinct elements of G. We conclude that G is uncountable.

We shall see in Section 3 that G embeds in the following inverse limit of free groups. Let F_n denote the subgroup of G generated by the first n elements $a_1, \ldots, a_n$. There is a natural projection $P_n : G \longrightarrow F_n$ like that used in the previous paragraph which collapses all but the first n circles to the base point. Therefore the groups F_n form a natural inverse limit into which G maps. One need only show, as we shall do in that section, that the limit map is injective.

Every finitely generated subgroup H of an inverse limit F of free groups F_α, for $\alpha \in A$, is free; we can see that as follows. Recall that the system A of indices is directed in the sense that, for each index α and β in A, there exists an index $\gamma \geq \alpha, \beta$. Further, if $\alpha \leq \beta$, then there is a homomorphism $f_\alpha^\beta : F_\beta \longrightarrow F_\alpha$, and all of these homomorphisms satisfy the natural commutativity and identity conditions. The limit group F is the subgroup of the direct product of the groups F_α consisting of those tuples, all of whose coordinate values are compatible under the homomorphisms f_α^β. Consequently there are natural homomorphisms $f_\alpha : F \longrightarrow F_\alpha$ for each α which commute with the homomorphisms f_α^β. Define H_α to be the image of H under f_α. Since each of the groups H_α is a free group of rank less than or equal to the minimal number of generators of H, there is a maximal rank n. Since the system A of indices is directed, since each of the homomorphisms $f_\alpha^\beta : H_\beta \longrightarrow H_\alpha$ is onto, and since homomorphisms of free groups can never increase rank, there is an index γ such that for each $\alpha \geq \gamma$, the rank of $H_\alpha = n$. Suppose therefore that $\gamma \leq \alpha \leq \beta$. Then the homomorphism $f_\alpha^\beta : H_\beta \longrightarrow H_\alpha$ is a surjective homomorphism between two free groups of rank n. Since finitely generated free groups are Hopfian, this homomorphism is an isomorphism. Let B be a finite subset of H which maps via f_γ onto a free basis for H_γ. It follows easily that the image coordinates in H_γ completely determine the entire group H, which must be free on the basis B.

We conclude from the preceding two paragraphs that every finitely generated subgroup of G is free.

We will now show that the group G is not free by showing that its abelianization, G^{ab}, is not free Abelian. This will follow from the fact that G^{ab} cannot be embedded in an inverse limit of free abelian groups. Finally, this follows from the fact that G^{ab} has an infinitely divisible element. We heard that such a fact is true from Mladen Bestvina who learned it from Geoffrey Mess. We trust that our construction is like that of Mess's. The explicit words we consider are very natural and have been considered before. See for example [H, Z2]. We shall give other, slightly harder but more informative proofs in Section 4. For our current purpose we simply need to find a single element of G whose image in the abelianization is nontrivial yet is divisible by infinitely many positive integers. To that end we use the commutators $c_n = [a_{2n+1}, a_{2n+2}]$, for $n = 0, 1, \ldots$. We describe a certain infinite product of such commutators in the following way. We begin our product with one occurrence of the commutator c_0: $w_0 = c_0$. Assume inductively that word w_{i-1} has been defined. Form word w_i by inserting two occurrences of c_i after each occurrence of c_{i-1}. Our final word w is then the natural limit of these finite approximations.

$$w_0 = c_0.$$
$$w_1 = c_0(c_1c_1).$$
$$w_2 = c_0(c_1[c_2c_2]c_1[c_2c_2]).$$
$$w_3 = c_0(c_1[c_2\{c_3c_3\}c_2\{c_3c_3\}]c_1[c_2\{c_3c_3\}c_2\{c_3c_3\}]).$$

We shall describe precisely in Section 3 how to realize w as an element of the Hawaiian earring group G. It is clear that in the abelianization of G we may make any finite number of word rearrangements and trivial cancellations. In particular, we may cancel any finite number of c_i's. By cancelling c_0, we see that all of the words from w_1 on become squares; hence w also becomes a square. By cancelling c_0 and c_1 we see that all of the words from w_2 on become fourth powers; hence w also becomes a fourth power. And so on. In Section 4 we shall see that a reduced word becomes trivial in the abelianization if and only if a finite number of word rearrangements and trivial cancellations makes a word trivial. (One has to be a bit careful about what a trivial cancellation means; we shall make that clear in Section 3.) It is easy to see, once the definitions have been made clear, that such a trivialization of w is impossible.

□

For another treatment of the next result, see [Z4].

Theorem 2.6. *Let E be the Hawaiian earring with base point $\mathbf{0}$, E_1 and E_2 two copies of E with corresponding base points $\mathbf{0}_1$ and $\mathbf{0}_2$, c_1E_1 and c_2E_2 cones over these copies, and $F = c_1E_1 \wedge c_2E_2$ the wedge product of the two cones joined at the two base points $\mathbf{0}_1$ and $\mathbf{0}_2$. Then c_1E_1 and c_2E_2 are contractible, but their wedge F is not. In fact, F has uncountable fundamental group and uncountable first homology group.*

Remark 2.6.1. It has been pointed out to us that, by the results of [E1], $H_1(F)$ is complete mod-U and contains the $\mathbb{Z}$-completion of $\mathbb{Z}$.

FIGURE 2. The doubled cone over the Hawaiian earring

Proof. Most of the proof we shall leave to the reader; we shall only show that the fundamental group of F is nontrivial. The stronger results follow in much the same manner.

Consider the loop that first traverses the element a_1 in E_1, then a_1 in E_2, then a_2 in E_1 followed by a_2 in E_2, ..., then a_n in E_1 followed by a_n in E_2, etc. We shall prove that this loop cannot bound a singular disk in F. Assume to the contrary that this loop bounds a singular disk $f : B^2 \longrightarrow F$ in F. Let p_n be a point other than the base point in circle n of E_1. Let q_n be a point other than the base point in circle n of E_2. Let I_n be the interval joining p_n to the midpoint of the join line which joins p_n to c_1. Let J_n be the interval joining q_n to the midpoint of the join line which joins q_n to c_2. Put the singular disk f in general position with respect to the union of the intervals I_n and J_n. The inverse image of I_n in B^2 must contain an arc A_n which starts on the boundary and has image which joins the ends of I_n. Similarly, the inverse image of J_n in B^2 must contain an arc B_n which starts on the boundary and has image which joins the ends of J_n. There exist subsequences whose interior endpoints converge in B^2, the A subsequence of ends converging to $p \in B^2$, the B subsequence of ends converging to $q \in B^2$. By continuity, p maps to the midpoint of the interval in c_1E_1 joining c_1 and $\mathbf{0}_1$. Similarly, q maps to the midpoint of the interval in c_2E_2 joining c_2 and $\mathbf{0}_2$. In particular, the points p and q are distinct. It follows that all of the interior endpoints of the A subsequence from some point on can be joined to one another in B^2 very near to p, hence certainly far away from the B subsequence, and similarly all of the interior endpoints of the B subsequence from some point on can be joined to one another in B^2 very near to q, hence certainly far away from the A subsequence. It follows easily that there are disjoint arcs in the disk B^2 whose endpoints lie in the boundary circle of B^2 and separate one another in that circle, a clear impossibility. We conclude that no such singular disk exists.

□

3. The big free groups BF(c)

3.1. **Transfinite words and the groups BF(c).** Let $G = \pi_1(E, \mathbf{0})$, where E is the Hawaiian earring. We shall use the properties discovered in Section 2.1 as motivation for the definition of the big free groups BF(c), where c is an arbitrary cardinal number. The geometric analysis of Section 2.1 suggests the following definitions.

Definition 3.1. Let A be an alphabet of arbitrary cardinality c, and let A^{-1} denote a formal inverse set for A. A *transfinite word* over A is a function $\alpha : S \longrightarrow A \cup A^{-1}$ satisfying the following two conditions:

$\alpha 1$: S is totally ordered.
$\alpha 2$: $\alpha^{-1}(a)$ is finite for each $a \in A \cup A^{-1}$.

We completely identify two transfinite words $\alpha_i : S_i \longrightarrow A \cup A^{-1}$ for $i = 0, 1$ if there is an order-preserving bijection $\phi : S_0 \longrightarrow S_1$ such that the following diagram commutes.

$$\begin{array}{ccc} S_0 & \xrightarrow{\alpha_0} & A \cup A^{-1} \\ \phi \downarrow & & \equiv\downarrow \\ S_1 & \xrightarrow{\alpha_1} & A \cup A^{-1} \end{array}$$

Theorem 3.2. *In the case where A is countably infinite as in section Section 2.1, if $f_0, f_1 : S^1 \longrightarrow E$ represent α_0 and α_1 respectively, then f_0 and f_1 represent the same element of $G = \pi_1(E)$.*

Proof. This follows from Theorem 2.3 as follows. Concatenate S_0 and the reverse $\bar{S}_1$ of S_1 to form $S_0\bar{S}_1$. Define the reverse $\bar{\alpha}_1 : \bar{S}_1 \longrightarrow A \cup A^{-1}$ by the formula $\bar{\alpha}_1(s) = \alpha_1(s)^{-1}$. Define $\alpha_0 \cdot \bar{\alpha}_1 : S_0\bar{S}_1 \longrightarrow A \cup A^{-1}$ by the formula

$$\alpha_0 \cdot \bar{\alpha}_1(s) = \begin{cases} \alpha_0(s) \text{ for } s \in S_0. \\ \bar{\alpha}_1(s) \text{ for } s \in \bar{S}_1. \end{cases}$$

There is an obvious pairing $* : S_0\bar{S}_1 \longrightarrow S_0\bar{S}_1$ which pairs $s \in S_0$ with $\phi(s) \in \bar{S}_1 = S_1$. Then the product path $f_0\bar{f}_1$ represents $\alpha_0\bar{\alpha}_1$ and is paired by $*$. Therefore $f_0\bar{f}_1$ is nullhomotopic by Theorem 2.3. □

Definition 3.3. If a and b are elements of the totally ordered set S then we define $[a, b]_S = \{s \in S \mid a \leq s \leq b\}$.

Definition 3.4. We say that a transfinite word $\alpha : S \longrightarrow A \cup A^{-1}$ admits a *cancellation* $*$ if there is a subset T of S and a pairing $* : T \longrightarrow T$ satisfying the following three conditions:

$1^\star$: $*$ is an involution of the set T.

$2^\star$: $*$ is *complete* in the sense that $[t,t^*]_S = [t,t^*]_T$ for every $t \in T$ and *noncrossing* in the sense that $[t,t^*]_T = ([t,t^*]_T)^*$ for every $t \in T$.

$3^\star$: $*$ is an *inverse pairing* in the sense that $\alpha(t^*) = \alpha(t)^{-1}$.

One might describe such a cancellation as a *complete noncrossing inverse pairing.* Note that $3^\star$ forces $*$ to be fixed–point–free.

We define $S/* = S \setminus T$ and $\alpha/* = \alpha|(S/*) : S/* \longrightarrow A \cup A^{-1}$. We say that $\alpha/*$ arises from α by cancellation, that α arises from $\alpha/*$ by expansion. We say that two transfinite words are equivalent if we can pass from one to the other by a finite number of expansions and cancellations.

FIGURE 3. A complete noncrossing inverse pairing

Theorem 3.5. *If A is countably infinite, if α_0, α_1 are two transfinite words, and if $f_0, f_1 : S^1 \longrightarrow E$ are representative functions, then α_0 and α_1 are equivalent if and only if f_0 and f_1 are homotopic in E.*

Proof. Suppose f_0 and f_1 are homotopic. Form the product $f_0 \cdot \bar{f}_1 : S^1 \longrightarrow E$, and extend this map to $F : B^2 \longrightarrow E$. Then the pairing $* : S_0\bar{S}_1 \longrightarrow S_0\bar{S}_1$ has pairs of three types: those joining points of S_0, those joining points of $\bar{S}_1 = S_1$, and those joining points of S_0 and $\bar{S}_1$. The first class defines a cancellation $*_0$ on α_0, the second a cancellation $*_1$ on α_1, the third an order isomorphism from $\alpha_0/*_0$ to $\alpha_1/*_1$. Thus $\alpha_0 \sim \alpha_0/*_0 \sim \alpha_1/*_1 \sim \alpha_1$.

Suppose conversely that $\alpha_0 \sim \alpha_1$, say by a simple cancellation $*$. Then there is a pairing $*' : S_0\bar{S}_1 \longrightarrow S_0\bar{S}_1$ which consists of two parts, namely the cancellation $*$ on S_0 and then the identity pairing between $S_0/* = (S_0 \setminus \text{domain}(*))$ and $\bar{S}_1 = S_1 = S_0/*$. Thus by Theorem 2.3, $f_0\bar{f}_1$ is nullhomotopic, or f_0 is homotopic to f_1. □

Definition 3.6 (BF(c)). We can multiply transfinite words by concatenation. This multiplication is clearly consistent with equivalence. Recall that if c is the cardinality of the alphabet A, we define BF(c) to be the set of equivalence classes of transfinite words on the alphabet A with this multiplication.

Definition 3.7 (Legal big products). We can extend this pairwise (and thus arbitrary finite) multiplication to a notion of infinite multiplication. Indeed, if $p : T \longrightarrow \mathrm{BF}(c)$ is a function from a totally ordered set to BF(c) so that there are choices of representative words $w_{p(t)}$ for each class $p(t)$ so that each $a \in A$ occurs in only finitely many of the $w_{p(t)}$ then p defines an element of BF(c),

$$\prod(p) = \prod_{t \in T} p(t).$$

The domain $D(p)$ of $\prod(p)$ is the disjoint union of the domains of the $w_{p(t)}$. The total ordering on $D(p)$ is defined by requiring that if $t_1 < t_2$ in T then every element of the domain of $w_{p(t_1)}$ be less than every element of the domain of $w_{p(t_2)}$. If $d \in D(p)$ then d is in the domain of $w_{p(t)}$ for some unique $t \in T$. Define $\prod(p)(d) = w_{p(t)}(d)$. Abusing notation a bit, replace $\prod(p)$ by the equivalence class it defines.

We leave the fact that $\prod(p)$ is well-defined by p to the reader. Such a $\prod(p)$ is called a *legal big product* of elements of BF(c).

For a different treatment of a similar result see [E2, Theorem A.1].

Theorem 3.8. *The set BF(c), with its multiplication induced by concatenation of transfinite words, is a group. We call it the* big free group on alphabet of cardinality c. *If A is countably infinite so that c is the countably infinite cardinal ω, then* BF(ω) *and $G = \pi_1(E)$ are isomorphic.*

Proof. The empty word acts as an identity. The reverse word acts as an inverse. Hence BF(c) is a group.

If A is countably infinite, define $\phi : G \longrightarrow \mathrm{BF}(\omega)$ as follows. Suppose $[f] \in G = \pi_1(E)$. The discussion preceding Theorem 2.2 showed that f gives rise naturally to a transfinite word α. By Theorem 3.5, the class of α is independent of the representative f of $[f]$ so that ϕ is well defined. The map ϕ is onto by Theorem 3.2. The map ϕ is clearly a homomorphism since it is well defined. The map ϕ is injective because of Theorem 3.5. Hence ϕ is an isomorphism. □

3.2. **Reduced words.** In this section we show that all of the big free groups BF(c) act much like the classical free groups in the sense that each element is represented by a unique reduced word; we say that a word α is reduced if it admits no nonempty cancellations. Clearly, every word α admits a maximal cancellation $*$ by Zorn's Lemma, and $\alpha/*$ is reduced. However, maximal cancellations are not unique (see Figure 4).

FIGURE 4. Maximal cancellations are not unique.

The following result is related to [E2, Theorem 1.4]

Theorem 3.9. *Each equivalence class of transfinite words contains exactly one reduced word.*

Remark 3.9.1. The proof can be reduced to the well-known classical case which recognizes that the reduced words on a finite alphabet form a tree, while an arbitrary word represents a position in that tree. We wish, however, to give another proof. We will use the topological separation properties of the plane so that this proof is conceptually deeper than the tree proof, nevertheless we find it enlightening.

Theorem 3.9′. *Suppose $\alpha : S \longrightarrow A \cup A^{-1}$ is a transfinite word and $*_0$ and $*_1$ are maximal cancellations. Then $\alpha/*_0$ and $\alpha/*_1$ are equivalent via an order-preserving bijection.*

Proof. Let $\sim$ be the smallest equivalence relation on S such that $x, y \in S$ are equivalent if either x and y are paired by $*_0$, or x and y are paired by $*_1$.

Construct a geometric realization $R([x])$ of each individual equivalence class $[x]$ as follows (see Figure 5). First note that each equiv-

FIGURE 5. Geometric Realization

alence class $[x]$ is finite since $*_0(y) = x$ implies $\alpha(y) = \alpha(x)^{-1}$, and similarly for $*_1$, while $\alpha^{-1}(a^{\pm 1})$ is finite for each $a \in A \cup A^{-1}$. Let $x_1 < x_2 < \cdots < x_k$ denote the elements of one equivalence class $[x]$. Take an order-preserving bijection onto a subset $x'_1 < x'_2 < \cdots < x'_k$ of the x -axis. Now join x'_i and x'_j by a semicircle in the lower half xy-plane if x'_i and x'_j are paired by $*_0$, by a semicircle in the upper half xy-plane if they are paired by $*_1$. If x'_i is not paired with anything by $*_0$, join x'_i to $-\infty$ by a vertical falling ray. If x'_i is not paired by $*_1$, join x'_i to $+\infty$ by a vertical rising ray. No semicircle or ray intersects another except at possible common endpoints because of condition $2^\star$. Hence the union of semicircles and rays is either a simple closed curve in the plane or a topological line. We denote this geometric realization of the equivalence class $[x]$ by $R[x]$. It depends precisely on the choice of $x'_1 < \cdots < x'_k$ in the x-axis. If we change this order preserving bijection into the x-axis, we change the size of the semicircles, but the combinatorial shape does not change.

We next note that any finite number of equivalence classes can be simultaneously realized. In fact, the union of any finitely many equivalence classes can simultaneously be mapped into the x-axis in an order preserving fashion and their respective geometric realizations will be disjoint, again by $2^\star$.

We next claim that, because of the maximality of $*_0$ and $*_1$, no class will be realized by a line with both ends at $-\infty$ or both ends at $+\infty$

(see Figure 6). Indeed, if there is such a line with both ends at $-\infty$,

FIGURE 6. This cannot happen

we show how to expand the *maximal* cancellation $*_0$, a contradiction. In such a class there are two points x_i' and x_j' joined to $-\infty$ by rays; that is, x_i and x_j are not identified with anything by $*_0$. However, they have inverse labels, $\alpha(x_i) = \alpha(x_j)^{-1}$ as one sees by a simple induction. We propose to identify x_i with x_j. Of course, we do this for every such pair. Suppose x is a point between x_i and x_j. We must show that x is paired by $*_0$ with a point y between x_i and x_j. We simultaneously realize the classes $[x_i] = [x_j]$ and $[x]$. If x is already identified to some y by $*_0$, then the semicircle joining x' to y' starts between the rays $x_i'(-\infty)$ and $x_j'(-\infty)$, hence ends there, so that y is between x_i' and x_j'. Otherwise $R[x]$ must contain the ray $x'(-\infty)$, which lies between the rays $x_i'(-\infty)$ and $x_j'(-\infty)$; consequently $R[x]$ must lie in the domain D of α bounded by $R[x_i] = R[x_j]$ which misses $+\infty$. Therefore $R[x]$ has both of its ends at $-\infty$, and both rays of $R[x]$ lie in D. Therefore there is another point y' of $R[x]$ joined to $-\infty$ in $R[x]$, y' lies between x_i' and x_j', and by agreement, the pair (x, y) is to be added to $*_0$. Thus our expansion of $*_0$ satisfies conditions $1^\star$, $2^\star$, and $3^\star$. Since $*_0$ was already maximal, we conclude that no such line exists. We conclude therefore that every $R[x]$ is either a single closed curve or a line from $-\infty$ to $+\infty$.

In order to show that $\alpha/*_0$ and $\alpha/*_1$ are equivalent via an order-preserving bijection, it suffices to show that, for any two letters $a, b \in A$, the four letters $a^{\pm 1}, b^{\pm 1}$ occur the same number of times in exactly the same order in $\alpha/*_0$ and $\alpha/*_1$ (see Figure 7).

FIGURE 7. Equivalent words

In order to verify that, we simultaneously realize the finitely many equivalence classes associated with $a^{\pm 1}$ and $b^{\pm 1}$. The result is a family of disjoint simple closed curves and lines in the plane, each line joining $-\infty$ to $+\infty$. It suffices to realize that it is precisely the lines that correspond to the uncancelled occurrences of $a^{\pm 1}$ and $b^{\pm 1}$ in $\alpha/*_0$ and $\alpha/*_1$. Each line has an associated letter, the same letter for $\alpha/*_0$ and $\alpha/*_1$. One reads the occurrences in $\alpha/*_0$ by reading the letters associated with the falling rays, the occurrences in $\alpha/*_1$ by reading the letters associated with rising rays. But rays are encountered in the same order one encounters lines in the separation properties of the plane. This completes the proof. □

3.3. **The big free group BF(c) as an inverse limit of free groups.** For each finite subset $A' \subset A$ there is a natural projection $\pi(A')$: $\mathrm{BF}(c) \longrightarrow F(A')$ onto the free group $F(A')$ with free basis A' which simply erases all letters other than those from $A' \cup A'^{-1}$. Thus there is a natural map from $\mathrm{BF}(c)$ into the inverse limit of these groups $F(A')$. The following lemma shows that the map is injective so that $\mathrm{BF}(c)$ is a subgroup of this inverse limit.

Theorem 3.10. *Suppose that $\alpha : S \longrightarrow A \cup A^{-1}$ has trivial projection $\pi(A')([\alpha]) \in F(A')$ for each finite subset A' of A. Then α is equivalent to the empty word.*

Exercise 3.11. Show that it is not enough to have $\pi(A')([\alpha])$ trivial for each two-element subset A' of A.

Proof. For each letter $a \in A$, let $C(a)$ denote the finitely many possible pairings that cancel the occurrences of a and a^{-1} in the word α. By hypothesis, using one–point subsets of A, each of the finite sets $C(a)$ is nonempty. Consider the product space C of these finite spaces. A point of C is therefore a collection of cancelling pairings, one for each letter and its inverse. Suppose a point $c \in C$ does not define a cancellation pairing for all of α. Then there are two letters, say a and b such that the cancellations $c(a)$ and $c(b)$ are incompatible. The cancellations $c(a)$ and $c(b)$ define an open subset of C consisting of all c' such that $c'(a) = c(a)$ and $c'(b) = c(b)$.

Now the theorem asserts that some $c \in C$ defines a cancellation pairing for all of α. If not, then the open sets described in the previous paragraph cover our compact space C, hence some finite subcollection does. But that means that every element of C is already self-incompatible on some fixed finite collection A' of letters, a contradiction to our hypothesis about the projection $\pi(A')$. □

4. Standard and Strong abelianization

4.1. **The commutator subgroup of BF(c).** We wish to characterize combinatorially those reduced words $\alpha : S \longrightarrow A \cup A^{-1}$ which represent elements of the commutator subgroup of BF(c). We shall do so in terms of a fixed-point-free involution $* : S \longrightarrow S : s \mapsto s^*$, which we fix for the rest of this section.

Definition 4.1. We say that identifications $s \leftrightarrow s^*$ and $t \leftrightarrow t^*$ are *parallel* if either $\{s, s^*\}$ lies in the closed interval with ends t and t^* or $\{t, t^*\}$ lies in the closed interval with ends s and s^*. If $s \leftrightarrow s^*$ and $t \leftrightarrow t^*$ are parallel, and if we have relabeled if necessary so that $s < s^*$, $t < t^*$, and $s \leq t$, then being parallel requires precisely that $s \leq t < t^* \leq s^*$. We say that two identifications $s \leftrightarrow s^*$ and $t \leftrightarrow t^*$ are $*$-*equivalent* if the following conditions are satisfied:

1 The pairs $s \leftrightarrow s^*$ and $t \leftrightarrow t^*$ are parallel. For convenience we may then assume that $s \leq t < t^* \leq s^*$.
2 Suppose $a \leftrightarrow a^*$ and $b \leftrightarrow b^*$ are identifications such that both sets $\{a, a^*\}$ and $\{b, b^*\}$ intersect the set $[s, t] \cup [t^*, s^*]$. Then $a \leftrightarrow a^*$ and $b \leftrightarrow b^*$ are parallel. In particular, $a \leftrightarrow a^*$ and $b \leftrightarrow b^*$ are parallel to $s \leftrightarrow s^*$ and $t \leftrightarrow t^*$ since the sets $\{s, s^*\}$ and $\{t, t^*\}$ intersect those intervals. If $a < a^*$, $b < b^*$, and $a \leq b$, then it follows that $s \leq a \leq b \leq t < t^* \leq b^* \leq a^* \leq s^*$.

Remark 4.1.1. We shall see in the following lemma that $*$-equivalence is an equivalence relation. Those identifications in a $*$-equivalence class can be pictured as a *rainbow*:

Figure 8. A rainbow

Think of the elements as residing in the x-axis in order preserving fashion; in the upper half-plane insert a semicircle joining each identified pair; then the semicircles fit together like a rainbow, and no semicircles are allowed to start in the one of the pots of gold where the rainbow touches the ground unless they belong to the given rainbow.

Lemma 4.2. *The relation of $*$-equivalence is an equivalence relation.*

Proof. The definitions have been stated in such a way that reflexivity and symmetry of the relation are obvious. As to transitivity, assume that $s \leftrightarrow s^*$ is $*$-equivalent to $t \leftrightarrow t^*$, and $t \leftrightarrow t^*$ to $u \leftrightarrow u^*$. We may assume that $s < s^*$, $t < t^*$, $u < u^*$ and $s \leq u$. We consider two cases.

Case 1: $s \leq t \leq u$. Suppose $a \leftrightarrow a^*$ and $b \leftrightarrow b^*$ given such that both sets $\{a, a^*\}$ and $\{b, b^*\}$ (with $a < a^*$, $b < b^*$ and $a \leq b$) intersect the set $[s, u] \cup [u^*, s^*]$. Then each intersects at least one of the unions $[s, t] \cup [t^*, s^*]$ and $[t, u] \cup [u^*, t^*]$. If they both intersect the same one of these unions, then they are parallel either by the hypothesis that $s \leftrightarrow s^*$ and $t \leftrightarrow t^*$ are $*$-equivalent or by the hypothesis that $t \leftrightarrow t^*$ and $u \leftrightarrow u^*$ are $*$-equivalent . Otherwise, we apply both hypotheses and discover that $s \leq a \leq t \leq b \leq u < u^* \leq b^* \leq t^* \leq a^* \leq s^*$, so that $a \leftrightarrow a^*$ and $b \leftrightarrow b^*$ are $*$-equivalent . We conclude in this case that $s \leftrightarrow s^*$ and $u \leftrightarrow u^*$ are $*$-equivalent .

Case 2: $t \notin [s, u]$. We lose no generality in assuming that $s \leq u < t$. Let $a \leftrightarrow a^*$ and $b \leftrightarrow b^*$ be given exactly as in Case 1. Then both of the sets $\{a, a^*\}$ and $\{b, b^*\}$ intersect the set $[s, t] \cup [t^*, s^*]$ because they intersect the smaller set $[s, u] \cup [u^*, s^*]$. Thus $a \leftrightarrow a^*$ and $b \leftrightarrow b^*$ are parallel by the hypothesis that $s \leftrightarrow s^*$ and $t \leftrightarrow t^*$ are $*$-equivalent . We therefore conclude in this case also that $s \leftrightarrow s^*$ and $u \leftrightarrow u^*$ are $*$-equivalent .

Thus $*$-equivalence is transitive as asserted. □

The following result is related to [E2, Lemma 4.11]. In particular that lemma proves the result for σ-words (in Eda's notation) but probably can be extended to the general case.

Theorem 4.3. *The reduced word $\alpha : S \longrightarrow A \cup A^{-1}$ represents an element of the commutator subgroup of $BF(c)$ if and only if there is a fixed-point-free involution $* : S \longrightarrow S$ satisfying the following two conditions:*

1. *For each $s \in S$, $\alpha(s^*) = \alpha(s)^{-1}$; and*
2. *There are only finitely many $*$-equivalence classes.*

Proof. First assume that α does represent an element of the commutator subgroup of $BF(c)$. In this case we must find the desired involution

$* : S \longrightarrow S$. Since α does represent an element of the commutator subgroup of BF(c), there exist reduced words v_1, w_1, ..., v_k, w_k such that the word $\alpha' = \alpha \cdot [v_1, w_1] \cdots [v_k, w_k]$ represents the trivial element of BF(c). (Here we are using the notation $[v, w]$ for the commutator $vwv^{-1}w^{-1}$.)

If S is the ordered set on which the function α is defined, and if S_i and T_i are the ordered sets on which the functions v_i and w_i are defined, then $S' = SS_1T_1\bar{S}_1\bar{T}_1 \cdots S_kT_k\bar{S}_k\bar{T}_k$ is the ordered set on which α' is defined. (Here we are using the notation that $\bar{S}$ is a copy of the ordered set S with order reversed; we call $\bar{S}$ the *reverse* of S.)

By Theorem 3.9, the word α' can be completely cancelled. Hence there is a fixed-point-free involution $*_0 : S' \longrightarrow S'$ which satisfies the cancellation laws of Section 2.1 and Section 3.1.

We let S'' denote the set $S' \setminus S$, the set on which the commutator product is defined. There is a natural fixed-point-free involution $*_1 : S'' \longrightarrow S''$ which, for each i, identifies corresponding points of S_i and $\bar{S}_i$ as well as corresponding points of T_i and $\bar{T}_i$. The involution $*_1$ is very decidedly not a cancellation. (Rather, it is an *Abelian* cancellation.)

We use the two involutions $*_0$ and $*_1$ to define an involution $* : S \longrightarrow S$ as follows. The involutions $*_0$ and $*_1$ can both be considered as relations on S' since the domain S'' of $*_1$ is a subset of the domain S' of $*_0$. Hence their union is a relation on S'. Extend this relation via transitivity to an equivalence relation on S'. Finally restrict this equivalence relation to S to obtain the desired relation $* : S \longrightarrow S$.

We can understand the relation $*$ geometrically in two steps as follows.

Step 1. Think of S as lying in an arc $A(S)$ on the boundary of a disk D_0. (Strictly speaking, this may be impossible since S may have more points than an arc contains. But for every finite subset of S this is certainly possible; and, if we wished, we could work with big disks and big arcs. Since the argument is really a finite argument, the reader can think about the proof in either manner.) Think of the other ordered sets S_i and T_i as also being embedded in arcs B_i and C_i, $\bar{S}_i$ and $\bar{T}_i$ embedded in arcs $\bar{B}_i$ and $\bar{C}_i$, in ∂D_0, so that, all together, we have the entire ordered set S' embedded in an order-preserving fashion in ∂D_0. By hypothesis, $*_0$ is a complete cancellation of α'; so it can be realized geometrically by a disjoint family of arcs spanning D_0, one arc for each pair $x \leftrightarrow x^*$ identified by $*_0$, the arc joining x and x^{*_0}.

Step 2. Identify the arcs B_i with the arcs $\bar{B}_i$, the arcs C_i with the arcs $\bar{C}_i$ in such a way that S_i is identified in the natural way with $\bar{S}_i$, T_i with $\bar{T}_i$. These identifications realize the involution $*_1$ and turn D_0 into a once-punctured compact connected orientable 2-manifold D

of genus k. The spanning arcs no longer end at any point of S'' but instead become disjoint connected 1-manifolds with boundary in S. Each contains precisely the points of a single equivalence class of our equivalence relation extending $*_0 \cup *_1$. Each of these connected 1-manifolds is, in fact, compact since its intersection with S' consists only of points whose images in $A \cup A^{-1}$ are the same or inverses of one another while a given letter is used only finitely often by hypothesis. Some of them may be simple closed curves, which we ignore. The others are arcs joining a point $s \in S$ to the point $s^* \in S$. All points of S are so joined. These arcs realize the involution $* : S \longrightarrow S$ geometrically.

It remains to show that the involution $*$ satisfies the two required conditions. First, we consider the easy condition 1:

An easy induction on the length of the arc shows that the endpoints of any of these arcs are mapped to inverse letters in $A \cup A^{-1}$.

Next, we establish the more difficult condition 2 by a geometric argument:

The key fact is that it is impossible to have arbitrarily many disjoint, nontrivial, nonparallel, properly embedded arcs in the once-punctured 2-manifold D (see the well-known lemma below). The arcs realizing the identifications of $*$ give us disjoint, properly embedded arcs. None of them is trivial in the sense that it bounds, together with a subarc of the arc $A(S)$, a disk in D; for otherwise we would find that $*$ defines a nontrivial cancellation in the reduced word α, namely the restriction of $*$ to that disk. Thus there are only finitely many parallel classes of identifying arcs. It remains only to show that the arcs of a parallel class are $*$-equivalent . Suppose A_i and A_j are identifying arcs that are parallel in D. That means that they are the sides of a topological rectangle R in D whose top and bottom are in the arc $A(S)$. Let A_k and A_m be any two identifying arcs which intersect the ends of R in $A(S)$. Since neither of them bounds, together with a subarc of the arc $A(S)$, a disk in D, they must both run from one end of R to the other. It follows that the identifications realized by A_k, and A_m are parallel. Thus, the identifications realized by A_i and A_j are $*$-equivalent .

Remark 4.3.1. It is interesting to note that the identifying arcs yield freely reduced words in the free fundamental group of D: One uses the fact that each of the words α, v_i, and w_i is reduced. Thus none of the identifying arcs of $*$ can be shortened combinatorially. Thus each defines a nontrivial reduced word in the free fundamental group of D which simply reads off the intersections with the arcs $B_i = \bar{B}_i$ and $C_i = \bar{C}_i$, respecting the side from which the arc approaches each arc B_i or C_i. Then two of the arcs are parallel if and only if they give precisely the same word.

Lemma 4.4. *Let D denote a compact connected orientable surface of genus k with one boundary component C. Choose a base point $x \in C$. Then there are at most $6k-1$ simple loops, based at x, disjoint except at x, that are nontrivial and nonparallel.*

Remark 4.4.1. The single point x corresponds to the entire arc $A(S)$ in the application. The loops correspond to the identification arcs. Two loops are said to be parallel if, together, they are the boundary of a disk in D. For the corresponding identification arcs, the condition can be stated as above in terms of a rectangle in D. Eda gets a similar estimate in [E2, lemma 4.16].

Proof of the lemma. Take any finite set of such loops (> 0) and cut the surface D along them. The result is a new surface D' which is not necessarily connected. Let K be a single component of D'. Then K is an orientable surface with a positive finite number of holes. The boundary of each hole is naturally divided into a finite number of open arcs by the various images of the base point x. One of those open arcs might be the image of the original boundary $C \setminus \{x\}$. Since C and the loops are nontrivial, K cannot be a disk with only one open arc in the boundary. Since no two of the loops are parallel, K cannot be a disk with exactly two open arcs in the boundary, each a copy of one of the loops. All other situations are possible. By examining the possibilities one-by-one in light of the classification theorem for surfaces, one easily sees that, by adding additional loops, it is possible to cut D' into disks, all but one having three loop-copies in the boundary, one having the copy of C and one additional loop-copy in its boundary. If n is the number of triangles, we can use this decomposition of D into one digon and n triangles in order to calculate the Euler characteristic $\chi(D) = 1-2k$ of D. Recall that there is only one vertex in D and that each edge except C lies on two faces:

$$1 - 2k = \chi(D) = 1 - [(3n+1)/2 + 1] + (n+1).$$

The total number of loops used in cutting D is clearly $N = (3n+1)/2$. If we solve the previous equation for N, we find, as required by the lemma, that

$$N = 6k - 1.$$

□

With the completion of the lemma, we have completed the first half of the proof of the theorem. The second half is elementary.

Lastly, assume that there is a fixed-point-free involution $*: S \longrightarrow S$ *satisfying conditions 1. and 2. of the theorem and having only finitely many* $*$*-equivalence classes.* Each equivalence class corresponds to a pair of words x and x^{-1}, where each is in fact a contiguous subword of α. The entire word α is thereby factored into finitely many such word pairs, in general nonadjacent. Modulo commutators, all of these words may be rearranged so that the words of the form x and x^{-1} are adjacent to one another. They then cancel in the most trivial of ways. We conclude that α represents an element of the commutator subgroup. □

4.2. **The Strong Abelianization and the Big Commutator Subgroup of the Hawaiian Earring Group.**

Notation 4.4.1. Throughout the rest of this section let G denote the fundamental group of the Hawaiian Earring group and let $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_i \supseteq \cdots$ denote the collection of subgroups of G induced by the subspaces of the Hawaiian Earring consisting of all but the outermost i circles. Let $P_i : G \longrightarrow F_i$ be the projection from the Hawaiian Earring group to the free group of rank i induced by the continuous map which takes the Hawaiian earring to itself by leaving the outermost i circles invariant and mapping all other circles to the basepoint.

As we have seen earlier, a reduced word w represents an element of the commutator subgroup, G', of the Hawaiian earring group G if and only if w can be factored into finitely many subwords $w = w_1 w_2 \dots w_n$ so that there is an inverse pairing σ on the w_i, (a permutation on n-letters of order two so that $w_\sigma(n) = {w_n}^{-1}$).

Definition 4.5 (Big commutator subgroup)**.** Let H be a subgroup of the Hawaiian earring group G. We say that a reduced word defines an element of the *big commutator subgroup,* $\overline{H}$*,* of H if and only if it can be factored as a *legal big product* (cf. Definition 3.7) of elements of H so that the factors have an inverse pairing. It is important to note that H depends not only on the isomorphism class of H but also on the embedding of H into the ambient group G.

Remark 4.5.1. In the case where H contains all of the a_i one can equivalently define $\overline{H}$ to consist of words which have an inverse pairing on their letters. More precisely, a word $w : S \longrightarrow A \cup A^{-1}$ in H, represents an element of $\overline{H}$ if there is an involution $\sigma : S \longrightarrow S$ so that $w(\sigma(s)) = w(s)^{-1}$ for all s in S.

Clearly big commutator subgroups enjoy a number of the elementary group theoretic properties of commutator subgroups. For instance if $H_1 < H_2$ then $\overline{H_1} < \overline{H_2}$

Definition 4.6 (abelianization)**.** As is standard, if H is a group we call the group H/H' the *abelianization* of H and denote it by H^{ab}.

Definition 4.7 (strong abelianization)**.** Correspondingly, if H is a subgroup of G, we define the *strong abelianization,* $\widehat{H}$*,* of H to be $H/\overline{H}$.

Remark 4.7.1. If g is an element of G then we may define $e_i(g)$ to be the exponent sum of a_i in g. This quantity is well-defined since it is invariant under word equivalence. Furthermore,the natural map, $\widehat{\ }: H \longrightarrow \widehat{H}$, has the property that the e_i are constant on fibres. Thus, the exponent sum, $e_i(\widehat{w})$, of a_i in an element $\widehat{w}$ in $\widehat{H}$ is a well–defined quantity, independent of H. It follows that the map

$$E_H : \widehat{H} \longrightarrow \prod_{i=n}^{\infty} \mathbb{Z}$$

defined by $E_H(\widehat{w}) = (e_1(\widehat{w}), e_2(\widehat{w}), \dots)$ is a homomorphism.

It is evident that E_G is an isomorphism, and $\widehat{G}$ is therefore torsion–free and, furthermore, has no infinitely divisible elements. It is interesting to note that though neither G nor $\widehat{G}$ contains infinitely divisible elements, G^{ab} does as we used in the proof (Theorem 2.5) that G is not a free group .

The strong Abelianization $\widehat{G} = G/\overline{G}$ of the Hawaiian earring group, G, though defined in terms of the alphabet A, has a group theoretic characterization which we now describe. First we need a definition.

Notation 4.7.1. Motivated by the fact that the elements of a group H which are perfect n-th powers generate the subgroup denoted as H^n, we will denote the group generated by those elements which can be written as arbitrarily large powers of elements of H(the infinitely divisible elements of H) as H^{∞}. The quotient H/H^{∞} can be denoted by $H^{1-\infty}$, or alternatively (as we shall do here) by $\mathcal{D}(\mathcal{H})$

Theorem 4.8. *The strong Abelianization,* $\widehat{G}$*, of the Hawaiian earring group,* G*, is the twice iterated quotient*

$$\mathcal{D}(\mathcal{D}(G/G')) = \mathcal{D}^2(G^{\mathrm{ab}}).$$

In other words, to obtain the strong Abelianization, $\widehat{G}$, of G, first Abelianize G, then quotient twice by the infinitely divisible elements.

Remark 4.8.1. This gives us a second proof that the Hawaiian Earring group, G, is not a free group. If it were free, then $\mathcal{D}^2(G^{\mathrm{ab}})$ would be equal to G^{ab} since the latter would be a free abelian group and thus have no infinitely divisible elements. However, it is a well-known fact that $\prod_{i=n}^{\infty} \mathbb{Z}$ is not a free abelian group. Another proof of this fact appears in the next section.

Remark 4.8.2. The uncountable rank free subgroup *Scatter*(c) of G described in Section 5 has $\widehat{G}$ (a group which is not free abelian) as its strong abelianization. However,

$$\mathcal{D}^2(\mathit{Scatter}(c)^{\mathrm{ab}}) = \mathit{Scatter}(c)^{\mathrm{ab}},$$

which is free abelian. It follows that the strong Abelianization of a subgroup of G very much depends on the word structure of the elements comprising the subgroup.

Exercise 4.9. All three quotients are necessary.

Proof. Choose D to be the subgroup of G containing G' so that

$$D/G' = (G/G')^{\infty}.$$

Thus,

$$G/D = \frac{G/G'}{D/G'} = \mathcal{D}(G/G').$$

Choose E to be the subgroup of G containing D so that

$$E/D = (G/D)^{\infty}.$$

Now,

$$G/E = \frac{G/D}{E/D} = \frac{\mathcal{D}(G/G')}{(G/D)^{\infty}} = \frac{\mathcal{D}(G/G')}{\mathcal{D}(G/G')^{\infty}} = \mathcal{D}(\mathcal{D}(G/G')) = \mathcal{D}^2(G^{\mathrm{ab}}).$$

Thus we need only show that $E = \overline{G}$.

First, it is straightforward that $E \subseteq \overline{G}$, since $G/\overline{G}$ is isomorphic to

$$\widehat{G} = \prod_{i=n}^{\infty} \mathbb{Z}$$

which has no infinitely divisible elements.

Now suppose $b \in \overline{G}$. As noted above, the exponent sums, $e_i(b)$, of the a_i are all zero. Hence, under the Abelianization $f : G \longrightarrow G/G'$, setting $a = f(b)$, we see easily that $a \in f(G_i)$ for all i.

The following argument will be used again with a slight modification in Theorem 4.14. For each $i \in \mathbb{N}$ choose $g_i \in G_i \cap f^{-1}(a)$. Define elements $w_{n,i}$ of G_i as follows,

$$\begin{aligned} w_{n,1} &= g_1 w_{n,2}^n \\ &\vdots \\ w_{n,i} &= g_i w_{n,i+1}^n \\ &\vdots \end{aligned}$$

The $w_{n,i}$ are well-defined by the above equations since they determine the projections $P_j(w_{n,i})$ for all i and j. Now,

$$\begin{aligned} f(w_{n,1}) &= f(g_1) + n \cdot f(w_{n,2}) = a + n \cdot (a + n \cdot f(w_{n,3})) \\ &= a + na + [n^2 \cdot f(w_{n,3})] \\ &\vdots \\ &= a + na + n^2 a + \cdots + n^{k-1} a + n^k \cdot f(w_{n,k+1}) \\ &= a \left(\frac{n^k - 1}{n-1} \right) + [n^k \cdot f(w_{n,k+1})], \ \forall\, n, k. \end{aligned}$$

Rearranging terms we obtain

$$\begin{aligned} (n-1) \cdot f(w_{n,1}) + a &= an^k + (n-1)n^k \cdot f(w_{n,k+1}) \\ &= n^k (a + (n-1) \cdot f(w_{n,k+1})), \ \forall\, n, k. \end{aligned}$$

Since k can be chosen freely, it follows for all n that

$$(n-1) \cdot f(w_{n,1}) + a$$

is an infinitely divisible element of G/G' and thus is contained in D/G'. Therefore

$$aD = [(1-n) \cdot f(w_{n,1})]D = (1-n) \cdot (f(w_{n,1})D)$$

for all n. It follows that aD is an infinitely divisible element of G/D and thus lies in E/D. Consequently $a \in E/G'$ and so $b \in E$. Whence $E \supseteq \overline{G}$ and the theorem is proven. □

4.3. **Homomorphisms from the Hawaiian Earring group to countable and Abelian Groups.** Since we are considering elements of the Hawaiian earring as infinite words it is only natural that we define a class of homomorphisms that respect this structure. This class will be the class of *infinitely multiplicative* homomorphisms. Note the relationship of the following definition to that of cancellation in

Definition 3.4. This relationship is to be expected since one can think of a homomorphism as a function between groups which respects finite cancellation in the domain.

Definition 4.10. If $f : G \longrightarrow L$ is a function from the Hawaiian earring group to a group L and

$$g = \prod_{s \in S} p(s)$$

is a legal big product of words (cf. Definition 3.7) in G then an *f–cancellation for g with domain T*, is a function $* : T \longrightarrow T$ for some $T \subseteq S$ which satisfies:

$1^\star$: $*$ is an involution of the set T.
$2^\star$: If $t \in T$ then $[t, t^*]_S = [t, t^*]_T = ([t, t^*]_T)^*$.
$3^\star$: $f(p(t^*)) = f(p(t))^{-1}$.

Definition 4.11 (Infinitely multiplicative homomorphism)**.** A homomorphism $f : G \longrightarrow L$ from the Hawaiian earring group to a group L is said to be an *infinitely multiplicative homomorphism* if whenever

$$g = \prod_{s \in S} p(s)$$

is a legal big product of words in G, and $*$ is an f–cancellation for g with domain T then

$$f(g) = f(\prod_{s \in S - T} p(s)).$$

The notion of infinitely multiplicative homomorphisms is first used in Corollary 4.13.

Theorem 4.12. *If $f : G \longrightarrow L$ is a homomorphism to a countable group then the decreasing sequence $f(G_0) \supseteq f(G_1) \supseteq \cdots$ of subgroups of L is eventually constant.*

Proof. We will proceed using a diagonalization argument. We may assume that f is surjective by replacing L by the image of f. By way of contradiction, assume $f(G_0) \supseteq f(G_1) \supseteq \cdots$ is not eventually constant. Then there exists a sequence of increasing natural numbers $(i_j)_{j=0}^{\infty}$ such that the index of $f(G_{i_{j+1}})$ in $f(G_{i_j})$ is greater than $j+1$. Also, since L is countable, we may enumerate its elements in a sequence $(l_i)_{i=0}^{\infty}$, and denote by the *subscript* of an element of L its position in this sequence.

We will construct a sequence of words, $(k_j)_{j=0}^{\infty}$, in G such that each k_j lies in G_{i_j} and so that the legal words

$$w_j = \prod_{i=0}^{j} k_j$$

satisfy the condition that every element of the coset $f(w_j)f(G_{i_{j+1}})$ has subscript greater than j.

Deferring the construction for a moment, we complete the argument by pointing out that the word

$$w = \prod_{i=0}^{\infty} k_j$$

is a legal word in G and that $f(w)$ is an element of $f(w_j)f(G_{i_{j+1}})$ for every j and thus has subscript larger than j for every natural number j, yielding a contradiction.

Choose $k_0 = 1_L$. If k_{j-1} (and thus also w_{j-1}) has been defined, we define k_j as follows. The index of $f(G_{i_{j+1}})$ in $f(G_{i_j})$ is greater than $j+1$ implies there is a set $\{r_1, r_2, \ldots, r_t\}$ of elements of G_{i_j} so that $\{f(r_1), f(r_2), \ldots, f(r_t)\}$ is a set of at least j+2 representatives of distinct left cosets of $f(G_{i_{j+1}})$ in $f(G_{i_j})$. It is evident that

$$\{f(w_{j-1}r_1), f(w_{j-1}r_2), \ldots, f(w_{j-1}r_t)\}$$

is also a set of representatives of distinct left cosets of $f(G_{i_{j+1}})$ in $f(G_{i_j})$. Since there only $j+1$ elements in L with subscript at most $j+1$, at least one of the cosets $f(w_{j-1}r_s)f(G_{i_{j+1}})$ contains no element whose subscript is less than $j+2$. Choose $k_j = r_s$. □

Corollary 4.13. *Let $f : G \longrightarrow L$ be a homomorphism to a countable group. Then the following are equivalent.*

1. *f is infinitely multiplicative.*
2. $$\bigcap_{i\in\mathbb{N}} f(G_i) = \{1_L\}.$$
3. *f factors through F_i via the projection P_i; in other words there is a homomorphism $k : F_i \longrightarrow L$ such that $f = k \circ P_i$.*

Proof. Suppose we have f such that (1) is true, but (2) is false and let l be an element of

$$\bigcap_{i\in\mathbb{N}} f(G_i) - \{1_L\}.$$

For each $i \in \mathbb{N}$ choose g_i an element of G_i such that $f(g_i) = l$. Now,

$$w = \prod_{i=1}^{\infty} g_i$$

is a legal word. However,

$$\begin{aligned} 1_L &= f(1_G) = f(w \cdot w^{-1}) \\ &= f(g_1 g_2 \cdots \bullet \cdots {g_2}^{-1} {g_1}^{-1}) \\ &= f(g_1) \cdot f(\overbrace{g_2 \, \overbrace{g_3 \cdots {g_2}^{-1}} \, {g_1}^{-1}}) \\ &= f(g_1) \cdot f(1_G) = l, \end{aligned}$$

which is a contradiction.

If we assume f satisfies (2), then, by Theorem 4.12, we have

$$\{1_L\} = \bigcap_{j \in \mathbb{N}} f(G_j) = f(G_i)$$

for some natural number i, which implies (3).

If the hypothesis of (3) is satisfied then (1) follows easily since the question of the infinite multiplicativity of f reduces to that of the (finite) multiplicativity of k.

□

The next three results are related to the notion of noncommutative slenderness. See Section 7 for details.

Theorem 4.14. *If A is an Abelian group with no infinitely divisible elements then every homomorphism $f : G \longrightarrow A$ has the property that*

$$\bigcap_{i \in \mathbb{N}} f(G_i) = 1_A.$$

Proof. Suppose that A contains a nonidentity element

$$a \in \bigcap_{i \in \mathbb{N}} f(G_i).$$

For each $i \in \mathbb{N}$ choose $g_i \in G_i \cap f^{-1}(a)$. Define elements $w_{n,i}$ of G as follows,

$$\begin{aligned} w_{n,1} &= g_1 w_{n,2}^n \\ &\vdots \\ w_{n,i} &= g_i w_{n,i+1}^n \\ &\vdots \end{aligned}$$

The $w_{n,i}$ are well-defined by the above equation because it determines the projections $P_j(w_{n,i})$ for all i and j. Now,

$$\begin{aligned} f(w_{n,1}) &= f(g_1) + n \cdot f(w_{n,2}) = a + n \cdot a + n \cdot f(w_{n,3})) \\ &= a + na + [n^2 \cdot f(w_{n,3})] \\ &\vdots \\ &= a + na + n^2 a + \cdots + n^{k-1} a + [n^k \cdot f(w_{n,k+1})] \\ &= a\left(\frac{n^k - 1}{n-1}\right) + n^k \cdot f(w_{n,k+1}),\ \forall\, n, k. \end{aligned}$$

So, by reorganizing, we have

$$\begin{aligned} (n-1) \cdot f(w_{n,1}) + a &= an^k + (n-1)n^k \cdot f(w_{n,k+1}) \\ &= n^k(a + (n-1) \cdot f(w_{n,k+1})),\ \forall\, n, k. \end{aligned}$$

Allowing k to take on arbitrarily large values and applying the hypothesis that A has no infinitely divisible elements it follows that

$$(n-1) \cdot f(w_{n,1}) + a = 0_A,$$

and thus for all n we have

$$a = (1-n) \cdot f(w_{n,1}).$$

Applying the hypothesis that A has no infinitely divisible elements once more we obtain $a = 0_A$. □

The last two results give rise to the following

Corollary 4.15. *If A is a countable Abelian group with no infinitely divisible elements then every homomorphism $f : G \longrightarrow A$ is infinitely multiplicative.*

Corollary 4.16. *The image of any homomorphism from the Hawaiian Earring group to a free Abelian group has finite rank.*

Proof. If there were a homomorphism from the Hawaiian Earring group onto a free Abelian group of infinite rank then there would be a homomorphism from the Hawaiian Earring group onto a free Abelian group of countably infinite rank. However by Corollary 4.15 any such map is infinitely multiplicative and thus by Corollary 4.13[(i),(iii)] factors through a map into a finite rank free group. This would contradict the fact that the image of this map has infinite rank. □

We now have yet a third proof of the following

Corollary 4.17. *The Hawaiian Earring group is not free.*

Proof. If the Hawaiian Earring group were free it would have infinite rank since it admits a map P_i onto a free group of every finite rank i. However, it cannot be an free group of infinite rank by the previous corollary. □

5. Classically-free subgroups of BF(c)

Throughout this section c will denote an infinite cardinal number and BF(c) the associated big free group on an alphabet A, $|A| = c$.

We have seen that the beg free group BF(c) is not free. In fact, by Corollary 4.16, we have seen that the Hawaiian earring group cannot be mapped onto a free Abelian group of infinite rank. Nevertheless we still use the topology of words to define a new subgroup *Scatter*(c) of BF(c) which is classically free and show that $|\,\mathit{Scatter}(c)| = \mathrm{BF}(c) = 2^c|$.

The group BF(c) has other large, classically-free subgroups. For example, Zastrow [Z2] shows that, in the case where $c = \aleph + 0$, the set of all words $w \in \mathrm{BF}(c)$ (the Hawaiian earring group) for which there is a finite bound b_w so that no letter may appear more than b_w times in w forms a classically-free subgroup of BF(c). He calls this group the *non-Abelian Specker group* after E. Specker. Specker proved that the group $\mathbb{Z}^{\mathbb{N}}$ is not free-Abelian but that, modulo the Continuum Hypothesis, the subgroup consisting of those sequences which have only finitely many different entries is free-Abelian. This subgroup of $\mathbb{Z}^{\mathbb{N}}$ is called the *Specker group* or *classical Specker group*.

Definition 5.1. Let $g \in \mathrm{BF}(c)$ and suppose g is represented by the (unique) reduced word $g : S(g) \to A \cup A^{-1}$. Define *space*($g$) to be the linearly-ordered space of Dedekind cuts in the linearly-ordered space $S(g)$. We shall see below that space(g) is compact and totally disconnected.

Definition 5.2. Recall that a space X is *scattered* if every nonempty subspace of X is has an isolated point. Define

$$\mathit{Scatter}(c) = \{g \in \mathrm{BF}(c) \mid \mathit{space}(g) \text{ is scattered}\}.$$

Theorem 5.3. *If c is an infinite cardinal, then* $|\,\mathrm{Scatter}(c)| = |BF(c)| = 2^c$*,* Scatter($c$) *is a subgroup of BF($c$) and* Scatter($c$) *is classically free.*

Outline of proof. The only claim which we can establish without preparation is the cardinality count, which we shall explain in a moment. That *Scatter*(c) is a subgroup requires an analysis of Dedekind cut spaces, an analysis which we undertake in the next section. The proof that *Scatter*(c) is classically free will proceed by transfinite induction on the depth of a scattered space and will appear in the following section. □

5.1. **Cardinality Count.** We construct injections $g : \mathbb{Z}^c \to \mathit{Scatter}(c)$ and $h : \mathrm{BF}(c) \to 2^c$ as follows. Identify c with the smallest ordinal of size c so that we can think of c as a well-ordered set. Identify 2^c with the subsets of c. If $u \in 2^c$, then, for each $x \in U$, insert a copy of the ordered set $\mathbb{Z}$ between x and $x+1$. The result is a linearly ordered set $c(U)$. Let $g(U))$ map $c(U)$ bijectively onto $A = A(c)$. Then $g(u))$ is a reduced word on the alphabet A. We claim that $g(u) \in \mathit{Scatter}(c)$ and that $g : 2^c \to \mathit{Scatter}(c)$ is injective.

In order to see that g is 1-1, it suffices to show that the linearly ordered set $c(u)$ determines U. Indeed, the first element x of U in c is the first limit of a *decreasing* sequence in $c(U)$. The next limit of an *increasing* sequence is $x+1$. Remove the intervening elements, and repeat transfinitely to recover U.

In order to see that $\text{space}(g(U)) = \text{Cut}(g(U))$ is scattered, we simply characterize the cuts in $c(U)$. There are two types, namely $(-\infty, x)$ with $x \in c(U) \cup \{\infty\}$ and $(-\infty, x]$ with $x \in U$. If $x \in c(U) \setminus c$, then $(-\infty, x)$ is clearly isolated. If we remove such cuts, the remaining cuts are well-ordered, one for each element of $c \setminus U$, two for each element of U, and one of the form $(-\infty, \infty)$. But, since every well-ordered set is scattered, it follows that $\text{Cut}(c(U))$ is scattered.

Let $g : S(g) \to A \cup A^{-1}$ be the reduced word representing an element of $\text{BF}(c)$. Since $g^{-1}(x)$ is finite for each $x \in A \cup A^{-1}$, we have $|S(g)| \leq c$. Hence, we lose no generality in assuming that $S(g) = c$, with a linear order perhaps different from that induced the the given well-ordering on c. We may identify the linear ordering as a relation on c, that is as a subset of $c \times c$ or an element of $2^{(c \times c)}$, with $S(g)$ as the domain of definition of that relation. With the map g we also associate a subset $A(g)$ of $(A \cup A^{-1}) \times \{0, 1, 2 \ldots\} \hookrightarrow c \times \aleph_0$ with $(x, i) \in A(g)$ if and only if $|g^{-1}(x)| \geq i$. We may then think of the function g as giving a bijection $\phi(g) : S(g) \to A(g)$ and note that this bijection completely determines g. That is, the function

$$g \mapsto (\phi(g), S(g), A(g))$$

is 1-1 on $\text{BF}(c)$. If $S(g)$ and $A(g))$ are given, then there are at most c^c choices for $\phi(g)$. There are at most $2^{(c \times c)}$ choices for $S(g)$, and at most $2^{(c \times \aleph_0)}$ choices for $A(g)$. But,

$$2^c = c^c \cdot 2^{(c \cdot c)} \cdot 2^{(c \cdot \aleph_0)}.$$

The existence of an injection $h : \text{BF}(c) \longrightarrow 2^c$ follows.

5.2. **Compact,connected linearly–ordered spaces.** We include here general results about compact, connected linearly–ordered spaces. While we do not use all of the results in this section elsewhere in the paper, we do use most of them. The remainder we will need in [CC2] to study the *big fundamental group* of *big Hawaiian* earrings and to show that they are big free groups.

We note that many of the results in this section can be found in the literature and texts on set theoretic topology. See [Eil] for example. We include their proofs here for the sake of completeness.

Let X be an arbitrary set and $<$ a linear order on X. For convenience in defining the linear-order topology, we prepend a point $-\infty$ to X which precedes all points of X, and we append a point $+\infty$ to X which follows all points of X. The linear-order topology on X has as basis the open intervals $(a, b) = \{x \in X \mid a < x < b\}$, where $a, b \in \{-\infty\} \cup X \cup \{+\infty\}$. In this subsection we recall the standard facts about compact, linearly-ordered spaces.

Exercise 5.4. Every linearly-ordered space is Hausdorff and normal. (The proof of regularity is easier than the proof of normality.)

5.2.1. *Subspaces.* A subset Y of a linearly-ordered space inherits both a subspace topology T_s and a linear order from which it derives a linear-order topology $T_<$.

Exercise 5.5. Show that the subspace topology and the linear-order topology of Y need not coincide.

Theorem 5.6. *If Y is compact, then the subspace topology T_s and the linear-order topology $T_<$ coincide.*

Proof. Since $T_< \subset T_s$, the identity map from (Y, T_s) to $(Y, T_<)$ is continuous. Therefore we have a continuous bijection from a compact Hausdorff space onto another Hausdorff space. It is therefore a homeomorphism. □

5.2.2. *Constructions.* We need to show the existence of arbitrarily large compact connected linearly-ordered spaces.

Example 5.6.1. Dedekind cut spaces. Let X be an arbitrary linearly-ordered set. A *Dedekind cut* in X is a subset C of X such that, if $x < c \in C$, then $x \in C$. We denote the set of all cuts in X by $\mathrm{Cut}(X)$. The set $\mathrm{Cut}(X)$ is naturally linearly-ordered by inclusion, with first cut $\emptyset$ and last cut X.

Theorem 5.7. *The linearly-ordered space* $\mathrm{Cut}(X)$ *is totally disconnected and compact.*

Proof. We see that the space $\mathrm{Cut}(X)$ is totally disconnected as follows. Let $c_1 < c_2$ be cuts. Let $x \in X$ be a point which lies in c_2 but not in c_1. There are two natural cuts associated with x, namely $(-\infty, x)$ and $(-\infty, x]$. The first clearly precedes the second and no point lies between them. Hence the space is separated between the two, hence also between c_1 and c_2. We conclude easily that $\mathrm{Cut}(X)$ is totally disconnected.

Compactness: Let $U = \{U_\alpha \mid \alpha \in A\}$ be an open cover of $\mathrm{Cut}(X)$ by basic open intervals $U_\alpha = (x_\alpha, y_\alpha)$. Any subset of $\mathrm{Cut}(X)$ which lies in the union of some finite subcollection of U is said to be *finitely covered*. Define $Y = \{c \in \mathrm{Cut}(X) \mid [\emptyset, c] \text{ is finitely covered}\}$. The first cut, the empty cut, is covered by a single element of the cover; hence the empty cut, the first element of $\mathrm{Cut}(X)$, lies in Y. The union c_0 of the cuts which are elements of Y is a cut. We first show that $c_0 \in Y$, then that $c_0 = X$, the last cut in X. Once we establish these two facts, we will conclude that U contains a finite subcover so that $\mathrm{Cut}(X)$ is compact.

$c_0 \in Y$: If $c_0 = \emptyset$, we already know that $c_0 \in Y$. Otherwise, there is an interval (a, b), with $a \geq \emptyset$, such that $c_0 \in (a, b)$, and (a, b) is covered by a single element of U. Since $a < c_0$, there is an element $x \in c_0 \setminus a$; and since $x \in c_0$, there exists $c \in Y$ such that $x \in c$. We now clearly have $a < (-\infty, x] \leq c$. Since $c \in Y$, the interval $(-\infty, c]$, which contains a, is finitely covered. Adding (a, b) to the finite cover, we see that $(-\infty, c_0]$ is also finitely covered, so that $c_0 \in Y$.

$c_0 = X$: Let (a, b) be an interval covered by a single element of U as in the previous paragraph. If $b = +\infty$, we obtain a finite cover of all of $(-\infty, c_0] \cup (a, b) = X$, and we are done. Otherwise b is a cut. If there is no cut between c_0 and b, then a finite cover of $(-\infty, c_0]$ and an element covering b together form a finite cover of $(-\infty, b]$, a contradiction to the fact that c_0 is the last element of Y. If there is a cut c between c_0 and b, then $(-\infty, c]$ is clearly finitely covered, which also contradicts the fact that c_0 is the last element of Y. The only noncontradictory possibility is that $c_0 = X$. □

Example 5.7.1. Well-ordered spaces form another excellent source of compact, linearly ordered spaces.

Theorem 5.8. *Suppose that X is a well-ordered space with a last point. Then X is compact.*

Proof. Again take an open cover by intervals. Again define a subspace Y consisting of those points $x \in X$ such that $(-\infty, x]$ are finitely covered. Let c_0 denote the least upper bound of the set Y.

First we show that $c_0 \in Y$: Clearly the first point of X is in Y. The point c_0 is covered by one interval (a, b), where we may assume that $-\infty < a$. Since $a < c_0$, $a \in Y$. Hence $(-\infty, a]$ is finitely covered. Adding the interval (a, b), we find that $(-\infty, c_0]$ is finitely covered. Hence $c_0 \in Y$.

Now c_0 is the last point of X – the proof is exactly like the last part of the proof of Theorem 5.7. □

Example 5.8.1. Connected spaces Both of our constructions above in Example 5.6.1 and Example 5.7.1 can be extended to create connected spaces. The following theorem succeeds for both cases.

Theorem 5.9. *Suppose that X is a compact, linearly-ordered space. Extend the space X and its linear order in the following way: if $x < y$ in X, and if there are no points of X between x and y, then add a real, open interval $(0, 1)_x$ with its natural order between x and y. The resulting space X_+ is compact, linearly-ordered, and connected.*

Proof. The space X_+ is linearly ordered by hypothesis. X_+ is compact: It is easy to prove that X_+ has a first point, a last point, and satisfies the least upper bound axiom. Then the proof of compactness follows the lines of proofs in Theorem 5.7 and Theorem 5.8

X_+ is connected: Let H and K be disjoint closed sets whose union is X_+, with the first point of X_+ in H. Let Y be the set of points x such that $(-\infty, x] \subset H$. Let c_0 be the least upper bound of the set Y. There is an interval about c_0 that lies either entirely in H or entirely in K. Since every interval about c_0 intersects H, this interval lies in H. The right hand part of the interval starting at c_0 cannot contain a point of K since the entire interval is in H. It also cannot contain a point c of H since otherwise $c \in Y$. Hence this half interval must be empty. It follows that c_0 cannot be an element of one of the added open intervals. Hence $c_0 \in X$. Since no interval was added immediately following c_0, the point c_0 must either be a limit point of successors in X, an impossibility, or c_0 must be the final point of X_+. We conclude that $K = \emptyset$ and that X_+ is connected.

□

Remark 5.9.1. What are the point pairs that have no points between them? In the case of the Dedekind cut space Cut(X), two cuts are adjacent if and only if they differ by precisely one point. Thus the intervals added in forming $\text{Cut}_+(X)$ from Cut(X) correspond precisely to the points of the original linearly ordered set X. In the case of the well-ordered space, every point has an immediate successor, hence is joined by an interval to that successor.

5.3. **Quotients.** It is very important in our proof that the big fundamental group is a set rather than just a class that we know how to pass from one compact linearly-ordered space to another via monotone quotient.

Definition 5.10. Monotone quotients. A *monotone decomposition* of a compact, linearly-ordered space X is an equivalence relation $\sim$ on X such that each equivalence class is either a single point or a closed interval in X. (Note that a closed interval in X is a set of the form $[x, y] = \{z \in X \mid x \leq z \leq y\}$ where $x < y \in X$; it may or may not be connected.) There is an associated identification space $X/{\sim}$ whose elements are the equivalence classes, and there is an associated projection map $\pi : X \longrightarrow X/{\sim}$ which takes each point to its equivalence class. A set U of $X/{\sim}$ is open if and only if $\pi^{-1}U$ is open in X. Note that $X/{\sim}$ also inherits a linear order from X so that $X/{\sim}$ can be endowed with a linear-order topology.

We have the following theorem analogous to Theorem 5.6.

Theorem 5.11. *Let X be a compact and linearly–ordered space. Let $\sim$ be a monotone decomposition of X. Let T_s denote the quotient topology and $T_<$ the linear-order topology on the set $X/{\sim}$. Then the topologies T_s and $T_<$ coincide.*

Proof. Suppose that we can show that the identity map from $(X/{\sim}, T_s)$ to $(X/{\sim}, T_<)$ is continuous. Then, since $T_<$ is Hausdorff, T_s is also; and, since T_s is compact, $T_<$ is also. Thus we will have a continuous bijection between compact Hausdorff spaces, necessarily a homeomorphism.

Consider therefore a subbasic open set of $(X/{\sim}, T_<)$ of the form $(-\infty, [b])$, where $[b]$ is the equivalence class of a point $b \in X$. Let b' be the first point of the closed interval $[b]$. Then $\pi^{-1}((-\infty, [b])) = (-\infty, b')$ which is open in X. Hence $(-\infty, [b])$ is open in the identification topology T_s. Similarly, the inverse of every other subbasic open set in $T_<$ is in T_s. Thus the identity map is continuous as required. □

Corollary 5.12. *If X is compact and linearly-ordered, and if $\sim$ is a monotone decomposition of X, then the space $X/{\sim}$ is compact and linearly-ordered, and the projection map $\pi : X \longrightarrow X/{\sim}$ is a closed map.*

Proof. The space $X/{\sim}$ is compact since the topology is the identification topology T_s. The space is linearly-ordered since $T_s = T_<$. The space is Hausdorff since linearly-ordered spaces are Hausdorff. The map π is closed because it is a continuous function from a compact space Hausdorff space to a Hausdorff space. □

5.3.1. *Quotients of Dedekind cut spaces.*

Theorem 5.13. *If X is a linearly ordered set and $Y \subset X$, then* $\mathrm{Cut}(Y)$ *is a monotone decomposition of* $\mathrm{Cut}(X)$*.*

Proof. Define $q : \mathrm{Cut}(X) \longrightarrow \mathrm{Cut}(Y)$ as follows. If C is a cut in X, then $q(C) = C \cap Y$ is a cut in Y. Clearly q is order preserving since $c_1 \leq c_2$ implies $c_1 \cap Y \leq c_2 \cap Y$.

The function q is onto: indeed, if C' is a cut in Y, define

$$c_1 = \{x \in X \mid \exists y \in C' \text{ such that } x \leq y\}$$

and

$$c_2 = \{x \in X \mid \text{ if } y \in Y \text{ and } y \leq x, \text{ then } y \in C'\}.$$

Then $q^{-1}(C') = [c_1, c_2]$, where, if $c_1 = c_2$, then $[c_1, c_2] = \{c_1\} = \{c_2\}$. We conclude that we can identify $\mathrm{Cut}(Y)$ with the equivalence classes of a monotone decomposition of $\mathrm{Cut}(X)$ in a natural way. Since $\mathrm{Cut}(Y)$ has the linear-order topology, it follows from Theorem 5.11 that the topologies coincide as well. □

It will be important to us to know the potential size of a monotone quotient space in terms of the size of a dense subset. Hence we consider the following definition.

Definition 5.14. We say that a nonempty subset D of the compact, linearly-ordered space X is *self-separating* in X if the following condition is satisfied: if $d_1 < d_2$ in D and if X intersects the open interval (d_1, d_2), then D intersects the open interval (d_1, d_2). If D is self-separating in X, we declare distinct points x and y of X to be equivalent, written $x \sim y$, if and only if at most one point of D intersects the closed interval from x to y.

Theorem 5.15. *Suppose D is self-separating in the compact linearly-ordered space X. Then the associated relation $\sim$ is a monotone decomposition of X. Furthermore, the image of D is dense in $X/\!\sim$.*

Proof. We first show that $\sim$ is an equivalence relation on X. Since $\sim$ is obviously reflexive and symmetric, we need only prove transitivity. Suppose therefore that distinct points x, y, z satisfy $x \sim y \sim z$. If z is between x and y, or x between y and z, then x and z are clearly equivalent. We therefore need only consider the case $x < y < z$. Suppose there are two points $d_1 < d_2$ of D in the closed interval $[x, z]$. Since $x \sim y$ and $y \sim z$, we must have $x \leq d_1 < y < d_2 \leq z$. But since D is self-separating in X and $d_1 < y < d_2$, there is a third point d of D in the interval $[d_1, d_2]$. But this third point contradicts either the equivalence $x \sim y$ or the equivalence $y \sim z$. We conclude that $[x, z]$ contains at most one point of D so that $x \sim z$. This argument proves the transitivity of $\sim$ and completes the proof that $\sim$ is an equivalence relation.

We next show that every equivalence class U which contains more than one point is in fact a closed interval. Let $a \in X$ be the greatest lower bound of U, $b \in X$ the least upper bound. It is easy to see that $(a, b) \subset U \subset [a, b]$; we simply need to show that $a, b \in U$. If $a \notin U$, then a is a limit point of U. Let $x \in U$. Since a and x are not equivalent, two points of D must lie in the interval $[a, x]$. For one of them, call it $d(x)$, we must have $a < d(x) \leq x < b$ so that $d(x) \in U$. It follows easily that D has infinite intersection with U, an obvious contradiction. Thus $a \in U$. Similarly, $b \in U$.

We conclude that $\sim$ is a monotone decomposition of X. It remains only to prove that the projection $\pi : X \longrightarrow X/\!\sim$ takes D to a dense subset of $X/\!\sim$. Suppose $\alpha < \beta < \gamma$ in $X/\!\sim \cup \{\pm\infty\}$. We need to show that $\pi(D)$ intersects (α, γ). This desired fact is obvious if $\alpha = -\infty$ and $\gamma = +\infty$ since $\pi(D) \neq \emptyset$. If $\alpha > -\infty$, let a be the last point of $\pi^{-1}(\alpha)$, b the first point of $\pi^{-1}(\beta)$. Since a and b are inequivalent, there are points d_1 and d_2 of D such that $a \leq d_1 < d_2 \leq b$. Thus $\pi(d_2) \in (\alpha, \gamma)$. If $\gamma < +\infty$, then we may proceed similarly. We conclude that $\pi(D)$ is dense. □

5.3.2. *The size of a linearly-ordered space.* We shall need to know that, up to homeomorphism, there is only a set's worth of spaces of a given size. If X is a set, define $P(X)$ to be the set of subsets of X. Inductively define $P^0(X) = X$ and $P^{n+1}(X) = P(P^n(X))$.

Lemma 5.16. *There are, up to homeomorphism, at most $|P^2(X)|$ topological spaces of cardinality $\leq |X|$. There are, up to homeomorphism, at most $|P^4(X)|$ regular Hausdorff spaces having a dense set of cardinality $\leq |X|$.*

Proof. If X' is a topological space of cardinality $\leq |X|$, then, up to homeomorphism, we may assume that $X' \subset X$. But this makes the topology on X' a set of subsets of X. That is, the topology is an element of $P^2(X)$. Hence there are, up to homeomorphism, at most $|P^2(X)|$ such topologies.

Let S be a regular Hausdorff space having as dense set a subset X' of X. Fix a point $p \in S$. Let $\{U_\alpha \mid \alpha \in A\}$ be the set of open sets containing p. Let $X_\alpha = U_\alpha \cap X'$. Note that X_α may equal X_β for $\alpha \neq \beta \in A$. However, X_α does determine the closure $\mathrm{cl}(U_\alpha)$ of U_α; and, since S is regular Hausdorff,

$$\bigcap_{\alpha \in A} \mathrm{cl}(U_\alpha) = \bigcap_{\alpha \in A} U_\alpha = \{p\}.$$

Thus the collection $\{X_\alpha \mid \alpha \in A\}$ determines p. But $\{X_\alpha \mid \alpha \in A\} \in P^2(X)$. We conclude that S has at most $|P^2(X)|$ points. Hence by the first assertion of the lemma, there are up to homeomorphism at most $|P^4(X)| = |P^2(P^2(X))|$ such spaces S. □

Among compact, connected, linearly-ordered spaces, the real interval $[0, 1]$ is easily characterized.

Theorem 5.17. *A compact, connected, linearly-ordered space X is homeomorphic with the real unit interval $[0, 1]$ if and only if it has a countably infinite dense set.*

Proof. The condition is clearly necessary. Suppose it is satisfied. Let $D = \{d_0, d_1, d_2, \dots\}$ be a countable dense set in X. We may assume that d_0 is the first point of X, d_1 the last point. Since X is connected, it easily follows that there is a point of D between each pair of points of D. It is easy to construct a linear-order-preserving bijection between the points of D and the rational points in $[0, 1]$, d_0 corresponding to 0 and d_1 corresponding to 1. Then one extends the correspondence to limit points by continuity. We leave the details to the reader. □

Lemma 5.18. *If S is a countable, linearly-ordered set, then* $\mathrm{Cut}(S)$ *embeds in the real line.*

Proof. Each element of S has two natural images in $\mathrm{Cut}(S)$ each of which is dense in $\mathrm{Cut}(S)$. If we add open intervals to form $\mathrm{Cut}_+(S)$, then we are adding only countably many intervals so that $\mathrm{Cut}_+(S)$ also has a countable dense set. It follows immediately from the previous lemma that $\mathrm{Cut}_+(S)$ is either a single point or a closed real metric interval. □

5.4. **Depth and Monotone Decompositions.**

Definition 5.19. We define the *transfinite derived sets* $X^{(\alpha)}$ of a space X as follows: $X^{(0)} = X$; $X^{(\alpha+1)}$ is the set of limit points of $X^{(\alpha)}$; if α is a limit ordinal, then

$$X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)}.$$

For brevity of notation we often write $X^{(1)}$ as X'.

Definition 5.20. We define a space to be *scattered* if each of its nonempty subspaces has an isolated point. It is easy to see that a space is scattered if and only if it has no perfect subspaces.

Lemma 5.21. *A space X is scattered if and only if there is an ordinal α such that $X^{(\alpha)} = \emptyset$. We call the smallest such ordinal α the* depth *of X and write $\alpha = \mathrm{depth}(X)$.*

Proof. If X is scattered, $X^{(\alpha+1)}$ is a proper subset of $X^{(\alpha)}$ for each ordinal α. Thus, for any ordinal α larger than the cardinality of X we must have $X^{(\alpha)} = \emptyset$.

Suppose α is an ordinal such that $X^{(\alpha)} = \emptyset$, and suppose Y is a subspace of X with no isolated points. Then $Y^{(\alpha)} \subseteq X^{(\alpha)} = \emptyset$. However, since Y has no isolated points, $Y = Y^{(1)} = \cdots = Y^{(\alpha)}$. It follows that Y is empty. □

Lemma 5.22. *If $f : K \to C$ is a continuous map of the compact Hausdorff space K onto the Hausdorff space C, then $C' \subseteq f(K')$. Furthermore $C^{(\alpha)} \subset f(K^{(\alpha)})$ for every ordinal α and consequently, if K is scattered then C is scattered with* $\mathrm{depth}(C) \leq \mathrm{depth}(K)$.

Proof. Suppose c is a point in $C \setminus f(K')$. We will show that $c \notin f(K)'$. Since $f(K')$ is compact and C is Hausdorff we may find neighborhoods U and V of c and $f(K')$ respectively which are disjoint. Then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint open sets in K containing $f^{-1}(c)$ and K' respectively. It follows that $f^{-1}(U)$ can contain no limit points of K and thus is closed. Also, since $f^{-1}(U)$ is disjoint from K', every point in $f^{-1}(U)$ is isolated and consequently every singleton subset of $f^{-1}(U)$ is open. Now, $\{\{k\} \mid k \in f^{-1}(U)\} \cup \{\{K - f^{-1}(U)\}\}$ is an open cover of K containing no proper subcover, so it must be that $f^{-1}(U)$ is finite. Hence U is a finite open neighborhood of c. Since C is Hausdorff, $\{c\}$ is open and thus $c \notin C'$. The rest of the result follows by an easy transfinite induction. □

Corollary 5.23. *If T is a subspace of linearly ordered space S and* $\mathrm{Cut}(S)$ *is scattered then* $\mathrm{Cut}(T)$ *is scattered with* $\mathrm{depth}(\mathrm{Cut}(T))) \leq \mathrm{depth}(\mathrm{Cut}(S))$.

Proof. We have a monotone decomposition map $\pi : \mathrm{Cut}(S) \longrightarrow \mathrm{Cut}(T)$ by Theorem 5.13. □

5.5. **Classically Free Subgroups.**

Definition 5.24. Each reduced word has a well-defined space, and a scattered space has a well-defined depth. Hence we may define the *depth* of an element whose space is scattered. Fix an ordinal $\alpha \geq 1$. Let $G_\alpha = \mathrm{BF}(c, \alpha)$ consist of those elements of $\mathrm{BF}(c)$ whose reduced words have scattered spaces of depth $\leq \alpha$.

Theorem 5.25 (Refinement of Theorem 5.3)**.** *The set G_α is a subgroup of $BF(c)$ which is free in the classical sense. We may choose bases B_α for the G_α such that, if $\alpha < \beta$, then $B_\alpha \subset B_\beta$.*

Proof of the refined Theorem 5.3. We first show that G_α is a subgroup. The empty word has a singleton as cut space, and individual letters have two-point cut spaces. These elements are therefore of depth 1. The cut space associated with a product of two words, before cancellation to reduce to normal form, is the wedge of the original two spaces, last point to first point. The depth of the wedge is the maximum of the depths of the factors. Reduction does not increase depth by Corollary 5.23 Thus G_α is closed under multiplication. Clearly a word and its reverse have the same depth. We conclude that G_α is closed under inversion. Therefore we may finally conclude that G_α is a subgroup of $\mathrm{BF}(c)$ and that G_1 is the classical free group on the alphabet A of cardinality c.

We proceed to prove that G_α is a classically free group by induction on α. We therefore assume that, for all $\beta < \alpha$, G_β is free with basis B_β and that, if $\beta < \gamma < \alpha$, then $B_\beta \subset B_\gamma$.

Case 1: α is a limit ordinal. Then we may take $B_\alpha = \cup_{\beta<\alpha} B_\beta$, and G_α is free on B_α.

Case 2: α is the immediate successor of β. We consider those elements g whose space $\mathrm{Cut}(S(g))$ has exactly one point in $\mathrm{Cut}(S(g))^\beta$ and that is the last point of $\mathrm{Cut}(S(g))$. We say that two such elements $g_1 : S_1 \longrightarrow A \cup A^{-1}$ and $g_2 : S_2 \longrightarrow A \cup A^{-1}$ *define the same germ at the terminal point* if there exist nonterminal points $s_1 \in S_1$ and $s_2 \in S_2$ such that $g_1|(s_1, \infty)$ and $g_2|(s_2, \infty)$ define the same word (recall the definition of word equality from Section 3.1) We call g_1 and g_2 equivalent if they define the same germ. This relation is an equivalence relation on our elements. We define B_α as follows: we choose one reduced word from each equivalence class, and add these germ representatives to the collection B_β. We must show that B_α is a free basis for G_α.

B_α generates G_α: Let $g : S(g) \longrightarrow A \cup A^{-1}$ be a reduced word in G_α. By definition, $\mathrm{Cut}(S(g))^\alpha = \emptyset$. If $\mathrm{Cut}(S(g))^\beta = \emptyset$, then $g \in G_\beta = \langle B_\beta \rangle \subset \langle B_\alpha \rangle$. If $\mathrm{Cut}(S(g))^\beta \neq \emptyset$, then $\mathrm{Cut}(S(g))^\beta$ is finite, for otherwise $\mathrm{Cut}(S(g))^\beta$ would have a limit point so that $\mathrm{Cut}(S(g))^\alpha \neq \emptyset$, a contradiction. Let $C_1 < C_2 < \cdots < C_k$ denote the cuts in $S(g)$ that lie in $\mathrm{Cut}(S(g))^\beta$. We obtain thereby a natural factorization of g into the finitely many words $g_1, g_2, \ldots, g_k, g_{k+1}$, where, for $i \leq k$,

$$g_i = g|(C_i \setminus C_{i-1} = \sigma)$$

and where

$$g_{k+1} = g|(S \setminus C_k = S_{k+1}).$$

For each g_i, $\mathrm{Cut}(S(g_i))^\beta$ is a subset of the two-point set of endpoints of $\mathrm{Cut}(S(g_i))$. We call an element of $\mathrm{Cut}(S(g_i))^\beta$ a β-limit point of $\mathrm{Cut}(S(g_i))$. If $\mathrm{Cut}(S(g_i))^\beta = \emptyset$, then $g_i \in G_\beta = \langle B_\beta \rangle \subset \langle B_\alpha \rangle$. If the terminal point of $\mathrm{Cut}(S(g_i))$ is a β–limit point of $\mathrm{Cut}(S(g_i))$, then g_i defines a germ at that terminal point for which we have chosen a representative $y_i \in B_\alpha \setminus B_\beta$. If it is not a β-limit point, we define $y_i = 1$. If the initial point of $\mathrm{Cut}(S(g_i))$ is a β-limit point, then g_i^{-1} has a β-limit point at its terminal endpoint for which we have chosen a representative $z_i \in B_\alpha \setminus B_\beta$. If it is not a β-limit point, we define $z_i = 1$. Then we consider the product $z_i g_i y_i^{-1}$. Then we can cancel away a neighborhood of each of the (at most 2) β-limit points. Hence $Cut(z_i g_i y_i^{-1})^{(\beta)} = \emptyset$ and $z_i g_i y_i^{-1} \in G_\beta$. Therefore the elements y_i, z_i, and $z_i g_i y_i^{-1}$ are all in $\langle B_\alpha \rangle$. Hence $g_i \in B_\alpha$. We conclude that $\langle B_\alpha \rangle = G_\alpha$.

The elements of B_α satisfy no nontrivial relation: Assume the contrary. Let $x_1 x_2 \cdots x_n$ be a cyclically reduced word in the letters of B_α of minimal length ≥ 1 which represents 1 in G_α. We then realize the product geometrically using the reduced-transfinite-word representation $x_i : \sigma \longrightarrow A \cup A^{-1}$, $x_1 x_2 \cdots x_n : S_1 S_2 \cdots S_n \longrightarrow A \cup A^{-1}$. Since $x_1 x_2 \cdots x_n = 1$ in G_α, there is a complete cancellation $* : S_1 S_2 \cdots S_n \longrightarrow S_1 S_2 \cdots S_n$. We shall analyze the cancellation $*$ to obtain a contradiction.

Since each x_i is reduced, each cancellation $s < s^*$ must span a join point where $\mathrm{Cut}(S_i)$ meets $\mathrm{Cut}(S_{i+1})$. Consequently, cancellations fall into finitely many equivalence classes, where cancellations $s_1 < s_1^*$ and $s_2 < s_2^*$ are called parallel if and only if the following condition is satisfied:

$$s_1 \leq s \leq s_2 \text{ iff } s_2^* \leq s^* \leq s_1^*.$$

The parallel class of a cancellation is completely determined by the join points it spans.

FIGURE 9. Parallel class of a cancellation

We now concentrate on a 1-sided β-limit point p. It must occur at a join point. All letters sufficiently near the point p must determine the same parallel class of identifications, and the image points converge to (determine) another 1-sided β-limit point. Thus we obtain a complete pairing of 1-sided β-limit points. These pairs satisfy the usual rules for an identification: that is, no two pairs separate one another. Since there are only finitely many, we take an innermost pair. We obtain thereby one of the following two pictures.

FIGURE 10. Two cases

We consider the two cases separately:

Case 1: there are no 1-sided limit points between the two, a and b, being paired. We conclude that the word between a and b is trivial and a product of x_i's, hence equals the entire word. Our outermost parallel class identifies the germ at b with the inverse of the germ at a, so that $x_1 x_2 \cdots x_n$ is not cyclically reduced, a contradiction.

Case 2: our parallel class identifies the left germ at c with the right germ at c. Then $x_1 x_2 \cdots x_n$ admits a cancellation, a contradiction.

We conclude that G_α is free on the generators B_α. Hence our induction is complete. $\square$

6. The big Cayley graph of BF(c)

For each reduced word $\alpha : S \longrightarrow A \cup A^{-1}$, we shall form a big interval I_α in the usual way: we first form the Dedekind cut space $\mathrm{Cut}(\alpha) = \mathrm{Cut}(S)$; we then insert real open intervals between adjacent points. The added open intervals all correspond to elements s of S, hence inherit a label $\alpha(s)$. We think of these open intervals as labeled and directed edges. Traversed in the opposite direction, we take the inverse label $\alpha(s)^{-1}$. We take the disjoint union of all of these big intervals I_α and then, for each pair, identify the largest initial segment on which all of the labels agree. They all agree on the empty cut. We take this cut as the *origin* or *identity* vertex of our big graph. Any two of the big intervals agree precisely on a closed initial segment. The result is the *big Cayley graph* $\Gamma(\mathrm{BF}(c))$. Branching occurs only at the cut points; we therefore call these points *vertices*. The open intervals we call *edges*.

Remark 6.0.1. *It is not clear which topology to choose for* $\Gamma(BF(c))$. In [BS2], Bogley and Sieradski describe a contractible metric topology for $\Gamma(\mathrm{BF}(c))$. We describe somewhat larger (nonmetrizable) alternatives which seem very natural to us.

What follows are two descriptions of topologies. Neither is a metric topology since neither is second countable. Both give the linear-order topology on the big intervals in $\Gamma(\mathrm{BF}(c))$. The big-metric topology is the most appealing, but there should be an end-compactification of both spaces, and it is most likely to be well–behaved with the finite-boundary topology. Perhaps with that topology and with an appropriate end compactification $\bar{\Gamma}(\mathrm{BF}(c))$, the action of $\mathrm{BF}(c)$ on $\bar{\Gamma}(\mathrm{BF}(c))$ is a convergence action.

6.1. **A finite boundary topology for $\Gamma(\mathbf{BF}(c))$.** Let x be a point of an open edge of $\Gamma(\mathrm{BF}(c))$. Then x divides each of the big intervals I_α which contains it into two halves, one half $J_\alpha(x)$ containing the origin, the other half $K_\alpha(x)$ missing the origin. Those big intervals I_β not containing x are likewise divided into two sets, namely $J_\beta(x) = I_\beta$ and $K_\beta(x) = \emptyset$. One first checks that these divisions are compatible with the identifications made in forming $\Gamma(\mathrm{BF}(c))$. Then one declares the two image sets $J(x) = \cup_\alpha J_\alpha(x)$ and $K(x) = \cup_\alpha K_\alpha(x)$ to be subbasic open sets for a topology on $\Gamma(\mathrm{BF}(c))$. Thus a basic open set has a finite boundary, a subset of the open edge set of $\Gamma(\mathrm{BF}(c))$.

6.2. **A big-metric topology for $\Gamma(\mathbf{BF}(c))$.** As metric parameter space we consider $\mathbb{R}^A_{\geq 0}$, the topological product of $c = |A|$–copies, $\mathbb{R}_{\geq 0}(a)$, of the positive reals $\mathbb{R}_{\geq 0}$, indexed by the alphabet A. We then define the *big distance* $d(x, y)$ between two points x and y of $\Gamma(\mathrm{BF}(c))$ as an element of $\mathbb{R}^A_{\geq 0}$ obtained as follows. We first note that there is a natural big interval between x and y. This is immediate if they are both contained in a single one of our original big intervals I_α. Otherwise, we find it as follows. We take one of our original intervals I_α which contains x and another of our original intervals I_β which contains y. We let c denote the last point on which I_α and I_β agree. Then c will be a vertex. The vertex c cuts I_α into two segments, one of which contains x. The vertex c cuts I_β into two segments, one of which contains y. One joins the appropriate two subsegments at c to obtain a big path $P(x, y)$ in $\Gamma(\mathrm{BF}(c))$ from x to y. This path will, for any letter $a \in A$, contain only finitely many open intervals (or fractions thereof) labeled with either a or a^{-1}. Then the a–coordinate of $d(x, y)$ counts how many such intervals with fractions of intervals (only occurring at the ends) measured by the real line metric applied to those copies of $(0, 1)$. This notion of distance turns $\Gamma(\mathrm{BF}(c))$ into what we call a *big metric space*. The definition of ϵ–neighborhood simply requires that one fix a neighborhood ϵ of 0 in $\mathbb{R}^A_{\geq 0}$.

Theorem 6.1. *With both the finite-boundary topology and the big-metric topology on* $\Gamma(BF(c))$, $\Gamma(BF(c))$ *is a Hausdorff space in which each space* I_α *is embedded. Also, between each pair of its points there is a unique big interval. The group* $BF(c)$ *acts homeomorphically on* $\Gamma(BF(c))$ *by what we can term* left multiplication. *This multiplication takes vertices to vertices and edges to edges. If* c *is an infinite cardinal, then there is no metric on* $\Gamma(BF(c))$ *which makes this action isometric.*

The proof is left to the reader.

Remark 6.1.1. Thanks to the theorem above, one can think of the big Cayley graph as a *big tree* on which the group acts *big isometrically*.

7. Noncommutatively Slender Groups

The referee has pointed out that our results in Section 4 were related to the theory of noncommutatively slender groups investigated by Eda and generously offered us the following description of those connections.

The concepts of cotorsion-freeness and slenderness of abelian groups are closely related to the results 4.14-16, and we collect here the information that is necessary to describe that connection. The basic reference is [F].

For the remainder of this section n will denote the set $\{n-1, \cdots, 0\}$, and A will denote an Abelian group.

Definition 7.1 (Cotorsion-freeness)**.** Let the abelian group A is said to be *cotorsion-free* if it has no nontrivial cotorsion subgroup.

Theorem 7.2. *The following are equivalent*

1. *A is cotorsion-free.*
2. *For any homomorphism $h : \mathbb{Z}^\omega \to A$, $\bigcap_{n<\omega} h(\mathbb{Z}^{\omega\setminus n}) = \{0\}$;*
3. *A is torsion-free and contains neither $\mathbb{Q}$ nor the p-adic integer group $\mathbb{J}_p$ for any prime p.*

Remark 7.2.1. The following argument is due to M. Dugas, R. Goebel, B. Wald et al. Since the referee did not find the appropriate reference, the proofs of some of the implications are indicated here.

Proof. (1) → (2)

Let A be a cotorsion-free group. Suppose that $\bigcap_{n<\omega} h(\mathbb{Z}^{\omega\setminus n}) \neq \{0\}$ for some $h : \mathbb{Z}^\omega \to A$. Let $0 \neq c \in \bigcap_{n<\omega} h(\mathbb{Z}^{\omega\setminus n})$ and pick $x_n \in \mathbb{Z}^{\omega\setminus n}$ so that $h(x_n) = c$ for each $n < \omega$. Let $\mathbf{e}_m$ be the element of $\mathbb{Z}^\omega$ defined by $\mathbf{e}_m(n) = \delta_{mn}$ for $m, n < \omega$. Since there exists a homomorphism $g : \mathbb{Z}^\omega \to \mathbb{Z}^\omega$ such that $g(\mathbf{e}_n) = x_n$ for each n, we may assume $h(\mathbf{e}_n) = c$ for all $n < \omega$.

An element of $\widehat{\mathbb{Z}}$, the $\mathbb{Z}$-completion of $\mathbb{Z}$, can be written as $\Sigma_{n<\omega} a(n)$ with $n!|a(n)$ for $a \in \mathbb{Z}^\omega$. Define $\overline{h} : \widehat{\mathbb{Z}} \to A$ by: $\overline{h}(\Sigma_{n<\omega} a(n)) = h(a)$. Then, $\overline{h}$ is a well-defined homomorphism. To see that $\overline{h}$ is indeed well-defined, let $\Sigma_{n<\omega} a(n) = \Sigma_{n<\omega} b(n)$ in $\widehat{\mathbb{Z}}$, i.e. $n!|\Sigma_{k=0}^{n-1} a(k) - \Sigma_{k=0}^{n-1} b(k)$ for each n. Since $U(A) = \{0\}$, it suffices to show $m!|h(a) - h(b)$ for each m.

Define a' by $a'(k) = 0$ for $k < m$ and $a'(k) = a(k)$ for $k \geq m$ and define b' similarly for b. Then, $m!|a'$ and $m!|b'$. Since $h(a) - h(b) = h(\Sigma_{k=0}^{m-1} a(k) - \Sigma_{k=0}^{n-1} b(k)) + h(a') - h(b') = (\Sigma_{k=0}^{m-1} a(k) - \Sigma_{k=0}^{n-1} b(k))c + h(a') - h(b')$, we get $m!|h(a) - h(b)$. Now, $\overline{h}(1) = h(\mathbf{e}_0) = c$ holds and we have a non-zero cotorsion subgroup of A, contradicting the assumption.

(1) → (3) There exists a non-zero homomorphism from $\mathbb{Z}^\omega / \oplus_\omega \mathbb{Z}$ to a non-trivial finite cyclic group, $\mathbb{Q}$ and also $\mathbb{J}_p$ for any prime p. Since a torsion-free cotorsion group is algebraically compact, we get the conclusion (3). □

Definition 7.3 (Slenderness)**.** A is said to be slender if, for each homomorphism $h : \mathbb{Z}^\omega \to A$, there exists n such that $h(\mathbb{Z}^{\omega\setminus n}) = \{0\}$.

Theorem 7.4. *The following are equivalent:*

1. *A is slender*
2. *A is cotorsion-free and does not contain $\mathbb{Z}^\omega$.*

Consequently, for a countable A, A is cotorsion-free if and only if A is slender. Also, if A has no infinitely divisible elements in the sense of Definition 2.4, then A is cotorsion-free because of the equivalence of (3) and (1) above. Now suppose that A is a countable group with no infinitely divisible elements. Then A is cotorsion-free and hence slender by the above. Theorem 3.3 of [E2] implies that A is noncommutatively slender and this immediately implies Corollary 4.15.

For an abelian group A, let $\mathcal{D}(A)$ be the group defined in Notation 4.7.1. For each algebraically compact group A, $\mathcal{D}^2(A) = \{0\}$. In order to explain the above, let us introduce a few notions, the first two of which are known, while the last one is suggested by the present paper.

Let $U(A) = \bigcap_{n<\omega} nA = \bigcap_{n<\omega} n!A$, called the Ulm subgroup, and

$$U_p(A) = \bigcap_{n<\omega} p^n A$$

for a prime p. Also, let

$$U_\infty(A) = \langle a : n|a \text{ for infinitely many } n \rangle.$$

Clearly, $U(A) \leq U_p(A) \leq U_\infty(A)$. The group A is said to be complete modulo U, if for any sequence $(a_n)_{n<\omega}$ such that $(n+1)!|a_{n+1} - a_n$ for each $n < \omega$, there exists a_∞ such that $(n+1)!|a_\infty - a_n$ for all $n < \omega$. Similarly, A is said to be complete modulo U_p, if for any sequence $(a_n)_{n<\omega}$ such that $p^{n+1}|a_{n+1} - a_n$ for each $n < \omega$, there exists a_∞ such that $p^{n+1}|a_\infty - a_n$ for all $n < \omega$. Now the following statements hold.

1. $U_\infty(\mathbb{J}_p) = \mathbb{J}_p$ for a prime p, where $\mathbb{J}_p$ is the group of p-adic integers;
2. If A is complete modulo U_p, then $U_\infty(A/U_p(A)) = A/U_p(A)$ holds and consequently $U_\infty(A/U_\infty(A)) = A/U_\infty(A)$ holds.
3. The statement (2) implies that $U_\infty(A/U_\infty(A)) = A/U_\infty(A)$ for each algebraically compact group A.

The proofs of statements (1) and (2) above proceed as follows. First recall the following facts on $\mathbb{J}_p$: (a) Each element of $\mathbb{J}_p$ is represented as a formal sum $\Sigma_{n<\omega} c_n p^n$ such that $0 \leq c_n \leq p-1$ for each n. (b) For each integer m which is not divisible by a prime p, there exists an element $x \in \mathbb{J}_p$ such that $mx = 1$ (in $\mathbb{J}_p$).

For each element $x \in \mathbb{J}_p$ and for each integer m which is not divisible by p, take an element $y \in \mathbb{J}_p$ such that $my = 1$. Then $x = m(xy)$ and this proves (1).

To show the statement (2), take $a \in A$ and take an integer m which is not divisible by p. There exists an element $\Sigma_{n<\omega} c_n p^n \in \mathbb{J}_p$ such that $m(\Sigma_{n<\omega} c_n p^n) = 1$. For each n, let $x_n = \Sigma_{i=0}^n c_i p^i a$. Then, $p^{n+1}|x_{n+1} - x_n$ for all $n < \omega$. Since A is complete modulo U_p, there exists $x_\infty \in A$ such that $p^{n+1}|x_\infty - x_n$ for all $n < \omega$. We claim that $a - mx_\infty \in U_p(A)$. Fix a positive integer k and take a large m_0 such that $p^k|m\Sigma_{i=0}^{m_0} c_i p^i - 1$. Put $n = \max\{k, m_0\}$. Then, $mx_\infty - a = m(x_\infty - x_n) + (mx_n - a) = m(x_\infty -$

$x_n) + (m(\Sigma_{i=0}^{m_0} c_i p^i) - 1)a$ is divided by p^k. Hence $a - mx_\infty \in U_p(A)$ and completes the proof of (2).

In Theorem 4.7 of [E2], it is shown that $\overline{G}/G'$ is complete modulo the Ulm subgroup. A similar proof to that of the theorem above shows that $\overline{G}/G'$ is complete module U_p and hence the above facts imply $\mathcal{D}^2(\overline{G}/G') = \{0\}$. Moreover, in a recent paper by Eda and Kawamura [EK], it is shown that $\overline{G}/G'$ is a torsion-free, algebraically compact group.

References

[BS1] W.A. Bogley, A.J. Sieradski, *Weighted Combinatorial Group Theory and Homotopy in Metric Complexes*, preprint.

[BS2] W.A. Bogley, A.J. Sieradski, *Universal Path Spaces*, preprint.

[CC2] J.W. Cannon, G.R. Conner, *The Big Fundamental Group, Big Hawaiian Earrings and the Big Free Groups*, preprint Brigham Young University, 1997.

[CC3] J.W. Cannon, G.R. Conner, *The Fundamental Groups of One–dimensional Spaces*, preprint Brigham Young University, 1997.

[dS] B. deSmit, *The fundamental group of the Hawaiian earring is not free.*, Internat. J. Algebra Comput. 2, no. 1, 33–37 (1992).

[E1] K. Eda, *The first integral singular homology groups of one point unions*, Quart. J. Math. Oxford **42** (1991), 443–456.

[E2] ______, *Free σ-products and noncommutatively slender groups*, J. Algebra **148** (1992), 243–263.

[EK] K. Eda and K. Kawamura, *The singular homology of the hawaiian earring*, preprint.

[Eil] S. Eilenberg, *Ordered topological spaces*, Amer. J. Math., **63** (1941), 39-45

[F] L. Fuchs, *Infinite Abelian Groups*, vol.1 (1970), Academic Press.

[G] H.B. Griffiths, *Infinite Products of Semi-groups and Local Connectivity*, Proc. London Math. Soc. (3), 455-456 (1956).

[H] G. Higman, *Unrestricted Free Products, and Varieties of Topological Groups*, Journ. London Math. Soc. 27, 73–81 (1952).

[MM] J.W. Morgan, I. Morrison, *A Van Kampen Theorem for Weak Joins*, Proc. London Math. Soc, 53, #3, 562–576 (1986).

[RLM] R.L. Moore, *Concerning upper semi-continuous collections of continua*, Trans. Amer. Math. Soc. **27** (1925), 416-428.

[S] A.J. Sieradski, *OMEGA-GROUPS: Group Theory for Wild Topology*,preprint.

[Z1] A. Zastrow, *All subsets of* $\mathbb{R}^2$ *are* $K(\pi, 1)$, preprint.

[Z2] A. Zastrow, *The Non-abelian Specker-Group is free*, preprint.

[Z3] A. Zastrow, *Construction of an infinitely generated group that is not a free product of surface groups and abelian groups but which acts freely on an* $\mathbb{R}$*-tree*, to appear Proc. R. Soc. Edinb.

[Z4] A. Zastrow, *The second van-Kampen theorem for topological spaces*, Topology and its Applications **59** (1994), 201–232.

Math Department, Brigham Young University, Provo, Ut. 84602, USA
E-mail address: cannon@math.byu.edu,conner@math.byu.edu