GEOMETRY AND DIAMETER BOUNDS OF DIRECTED CAYLEY GRAPHS OF ABELIAN GROUPS

CHARLES M. FIDUCCIA, RODNEY W. FORCADE, AND JENNIFER S. ZITO

Abstract. Many popular interconnection network topologies, such as hypercubes and toroidal meshes, are based on Cayley graphs of Abelian groups. The symmetry and algebraic structure of these graphs result in many nice physical properties of the network concerning layout, routing algorithms, and load balancing. There has been interest in low-diameter Abelian-Cayley graphs because of their smaller communication delay and reduced congestion. For any fixed number of nodes $n$, and any fixed out-degree $k$, we are interested in how small the diameter of directed Cayley graphs of Abelian groups can be and what these low-diameter graphs look like. We give an upper bound of $n \leq \frac{16k^3}{3}$ for the size of directed Abelian-Cayley graphs with $k = 3$ and diameter $d$, correcting a previously published result by Hsu and Jia [SIAM J. Discrete Math., 7 (1994), pp. 57–71].

Our method is based on translational tiling techniques and is a generalization of Wong and Coppersmith's method for $k = 2$ [J. Assoc. Comput. Mach., 21 (1974), pp. 392–402]. Moreover, our method works for all Abelian groups, not just the cyclic case. For $k = 3$ we give computational results for the largest Abelian-Cayley graph as a function of diameter. When $n = 8d^3$, for integer $m$, there is a network with $n = \left\lfloor \frac{4d^3}{11m} \right\rfloor$ whose diameter is approximately three-fourths of that of a three-dimensional toroidal cube.

Key words. Cayley graphs, Abelian groups, diameter, network topologies, translational tilings, degree-diameter problems

AMS subject classifications. 05C25, 05C12, 20F05, 68R10, 90B12

1. Introduction and problem history. The directed Cayley graph associated with an Abelian group $G$ and an edge generating set $E \subset G$ has the elements of $G$ as its vertices and directed edges from each $g$ to all vertices $g' = g \pm h$, where $h \in E$. The out-degree $k$ of the graph, as well as its in-degree, is thus equal to the cardinality of $E$.

The process of moving through a Cayley graph can be geometrically represented by associating each generator of the group with an orthogonal direction (i.e., a unit vector) in $k$-dimensional space, starting with the group identity at the origin. Thus each edge-generator of the Cayley graph corresponds to a direction in the lattice. Given the Cayley graph of an Abelian group $G$, with $n$ elements and edge-generating list $E = \{g_1, g_2, \ldots, g_k\}$, we may define a mapping $\phi$ from the $k$-dimensional lattice of integers to $G$. The map $\phi : \mathbb{Z}^k \to G$ takes the lattice point with coordinates $(x_1, x_2, \ldots, x_k)$ to the group element $g = x_1g_1 + x_2g_2 + \cdots + x_kg_k$ and is a group homomorphism.

Now we do a breadth-first search in the positive orthant of the lattice $\mathbb{Z}^k$, starting at the origin, until we find all the elements of the group $G$. We do the search in shortlex order with respect to the $k$ dimensions. Shortlex ordering for lattice points is defined as follows: the lattice point $\vec{x} = (x_1, x_2, \ldots, x_k)$ precedes the lattice point $\vec{y} = (y_1, y_2, \ldots, y_k)$ if either $\vec{x}$ has smaller Manhattan distance from the origin ($\sum_{i=1}^{k} |x_i| < \sum_{i=1}^{k} |y_i|$)
\[ \Sigma_{i=1}^{k} y_i \] or \( \vec{x} \) and \( \vec{y} \) have equal Manhattan distance from the origin and \( \vec{x} \) precedes \( \vec{y} \) lexicographically. If \( g \in G \), let \( \text{ShortLex}(g) \) be the first lattice point \( \vec{x}' \) found during the short-lex search such that \( \phi(\vec{x}') = g \). Now let \( L = \{ \text{ShortLex}(g) : g \in G \} \). The set \( L \) consists of the lattice locations for the elements of \( G \) as they are first traversed in the short-lex search. Because \( \phi \) is a one-to-one map of \( L \) onto \( G \), the elements of \( L \) form a complete set of coset representatives (a transversal) for the subgroup \( H = \ker(\phi) \) (the kernel). In other words, each element \( z \) of \( Z^k \) is uniquely represented as a sum \( z = x + h \), where \( x \in L \) and \( h \in H \), which is another way of saying that \( L \) tiles \( Z^k \) via the translation group \( H \), that is, \( Z^k = L + H \).

Note that the diameter of \( G \) corresponds to the greatest Manhattan distance of any point in \( L \) from the origin. We will accordingly call it the diameter of \( L \).

Now we form a solid tile from our set of lattice points \( L \) by taking the union (in \( R^k \))

\[ T = L + [0, 1)^k = \bigcup_{x \in L} (x + [0, 1)^k) \]

of the unit \( k \)-cubes located at the lattice points of \( L \). This forms a \( k \)-dimensional connected shape which we will call the Cayley tile at the origin. An example is given in Figure 1.

Note that a tiling of \( Z^k \) by \( L \), with the translation group \( H \), corresponds to a tiling of \( R^k \) by \( T \), using the same translation group. Note also that, if \( d \) is the diameter of \( L \), as defined above, then \( d + k \) is the greatest Manhattan distance of any point of the closure of \( T \) from the origin. We will call \( D = d + k \) the diameter of \( T \). (To avoid confusion, we will consistently use \( D \) for the latter (solid) diameter and \( d \) for the former (lattice) diameter.)

The following example is the Cayley tile created by taking the cyclic group \( Z_{84} \), with edge-generating list \( E = [2, 9, 35] \). This tile was found via computer search by one of our summer students, Wei-Hwa Huang, in 1993 and independently by Randall Dougherty and Vance Faber [4]. Readers interested in Cayley graphs as network topologies may also like to see some of the seminal papers in the area [1, 2, 3, 6, 7, 9].

**2. Necessary condition on the three-dimensional tiles.** The Cayley tile \( T \) has several useful properties. If \( x = (x_1, x_2, \ldots, x_k) \) and \( y = (y_1, y_2, \ldots, y_k) \) are elements of \( R^k \), we will say \( x \preceq y \) when \( x_i \leq y_i \) for all \( i \). Let \( (0) \) denote the origin in \( R^k \). Let \( \hat{e}_i \) denote the \( i \)th unit vector in \( R^k \). Then a notch in \( T \) is a point \( x \) in the
boundary of $T$ so that, for $\epsilon > 0$, $x + \epsilon \hat{e}_i$ is also in the boundary of $T$ for each $i$, but $x + \sum_{i=1}^{k} \epsilon \hat{e}_i$ is not in the boundary of $T$, for any $\epsilon > 0$ (for example, the point $(2, 5, 1)$ in Figure 2). In other words, a notch is a place where it looks like a translation of the first orthant has been cut out of the tile. By the silhouette of a Cayley tile $T$, we mean the set of points $y$, with at most one nonzero coordinate, such that $y \preceq x$ for some point $x \in T$ (in other words, the projection of $T$ into the coordinate hyperplanes). See Figure 3 for an example.

**Theorem 2.1.** Every Cayley tile $T$ has the following properties:

1. If $x \in T$ and $(0) \preceq y \preceq x$, then $y \in T$.
2. $T$ has at most one notch.
3. A Cayley tile $T$ is uniquely determined by its silhouette and, if it has a notch, the coordinates of its notch.

**Proof.**

1. First, observe that $L = \text{ShortLex}(G)$, as defined above, has this property as a subset of $\mathbb{Z}^k$. For if $(0) \preceq y \preceq x$, with $x \in L$ then, either $y - x$, or $y$ is closer (in Manhattan distance) to the origin. In the former case, there is nothing
to prove. In the latter case, let \( \phi(x) - g \) and \( \phi(y) - h \). Then \( y \notin L \) means there exists \( y' \) earlier in short-lex order than \( y \), with \( \phi(y') - h \). But then \( x' - y' + (x - y) \) is earlier in short-lex order than \( x \) and \( \phi(x') - h + (g - h) - g \), contradicting \( x = \text{ShortLex}(y) \). Second, observe that the \( k \)-cube \([0, 1]^k\) has this property, as a subset of \( R^k \) (trivially). Putting these two observations together implies that \( T - L + [0, 1]^k \) also has the property.

2. Suppose \( T \) has at least two notches. Call them \( x \) and \( y \). Clearly they must be elements of \( Z^k \). As already observed, the neighborhood of a notch looks like a neighborhood of the origin in the complement of the first octant. Thus, in the translational tiling by \( T \), the only way to “fill” that point is with a translation of the origin itself. Thus both notches are images of the identity of \( G \) and so is their difference \( x - y \). Since they are notches, each point is immediately above a point of \( Z^k \) which is in the tile (both \( T \) and \( L \)). Simply subtract one from the first coordinate of \( x \) and \( y \) to get \( x' \) and \( y' \), respectively. Then \( x' - y' = x - y \), so \( x' \) and \( y' \) have the same \( \phi \)-image, contradicting the definition of \( L \).

3. If \( x \in T \), then \( \pi_i(x) \) is in the silhouette, for each projection \( \pi_i \). Furthermore, if there is a notch, \( y \), then \( y \notin x \). Conversely, we show that if a point \( x \) has these two properties, then it is an element of \( T \). Let \( x \) be an element of \( R^k \) with every projection \( \pi_i(x) \) in the silhouette of \( T \), and \( x \notin T \). Then, decreasing the coordinates of \( x \), successively, there is a point \( y \leq x \) with \( y \notin T \); but every point \( z \) with \( z \leq y \) is in \( T \). Clearly, \( y \) is a notch. Thus, if \( x \) has all its projections in the silhouette and is not preceded by a notch, then \( x \in T \). Thus we have shown that the silhouette and (if it exists) the notch of \( T \) entirely determine which points are in \( T \). ⌜

These are necessary, but not sufficient, conditions for a shape to be a Cayley tile. We note that Wong and Coppersmith [9] proved the one-notch result (2) for the two-dimensional case with cyclic groups and one generator equal to the identity.

3. Improved diameter bounds. Given a solidified Cayley tile \( T \), with solid diameter \( D \) and volume \( V \), we will show that there exists a shape \( S \) with the same diameter and which, although it is not a superset of \( T \), necessarily has greater volume than \( T \). The volume of \( S \) will be our bound for that of \( T \). To clarify our argument, however, we do it first with a simplifying assumption—that our tile \( T \) has no notch. This will give a bound on volume which holds only for tiles with no notch and which is too small to be proved in the general case.

No notch argument. If \( T \) has no notch, it is entirely determined by its silhouette. This means that every point outside of \( T \) in the first octant is connected to (at least) one of the coordinate planes by a perpendicular to that coordinate plane. This perpendicular does not intersect \( T \). We may classify a point \( p \notin T \) in the first octant, as being of type \( x \), \( y \), or \( z \), according to whether there is a line through \( p \) parallel to the \( x \)-axis, \( y \)-axis, or \( z \)-axis, respectively, which does not intersect \( T \). Note that \( p \) may be, simultaneously, of more than one type.

**Lemma 3.1.** If \( p \leq p_1 \), then \( p_1 \) has (at least) the same type(s) as \( p \).

**Proof.** Suppose (for example) that \( p \) has type \( x \). Then the segment \( p_1 \), joining \( p \) perpendicularly to the nearest point \( q \) in the \( yz \) plane, does not intersect \( T \) (thus \( q \notin T \)). But, for every point \( u \) on the segment \( p_1 q_1 \) joining \( p_1 \) perpendicularly to the point \( q_1 \) in the \( yz \) plane, \( q \leq u \) so \( u \notin T \). Thus \( p_1 \) also has type \( x \). Analogous arguments work for the other two types. ⌜

Let \( P \) be the plane defined by \( x + y + z = D \). Let \( O \) denote the origin. Let \( A_1 \) —
\((D,0,0), A_2 - (0,D,0), \text{ and } A_3 - (0,0,D)\). Then \(T\) is a subset of the tetrahedron \(OA_1A_2A_3\), enclosed by the three coordinate planes and \(P\). Let \(Q\) denote the triangle \(A_1A_2A_3\) (including its interior area).

Let \(\Gamma_x\) denote the subset of \(Q\) comprising all points of type \(x\). Let \(\Gamma_y\) denote the subset of \(Q\) comprising all points of \(Q\) which are not in \(\Gamma_x\) and which are of type \(y\). Let \(\Gamma_z\) be the set of all points of \(Q\) which are not in \(\Gamma_x \cup \Gamma_y\) and which are of type \(z\). By our assumption (no notch) \(Q - \Gamma_x \cup \Gamma_y \cup \Gamma_z\). We have also arranged that this be a disjoint union.

From each point \(p\) in \(\Gamma_x\), one may drop a segment \(p_{\Gamma_x}\), parallel to the \(z\)-axis, to a point \(q_p\) in the \(yz\) plane, without intersecting \(T\). The union of all such segments \(p_{\Gamma_x}\) (\(p \in \Gamma_x\)) forms a solid \(G_x\) in the complement of \(T\). Similarly, from each point in \(\Gamma_y\) we drop a perpendicular segment to the \(xz\) plane and let \(G_y\) be the union of those segments, and from each point in \(\Gamma_z\) drop a perpendicular segment to the \(xy\) plane, thus forming \(G_z\).

**Lemma 3.2.** The sets \(G_x\), \(G_y\), and \(G_z\) are disjoint.

**Proof.** Suppose \(r \in G_x \cap G_y\). Then \(r\) is on a line segment \(p_1q_1\) from a point \(p_1 \in \Gamma_x\) to the \(yz\) plane and \(r\) is also on a segment \(p_2q_2\) from a point \(p_2 \in \Gamma_y\) to the \(xz\) plane. Clearly, \(r\) has both types and \(r \neq p_1\) and \(r \neq p_2\). By Lemma 1, \(p_2\) is therefore of type \(x\) and should therefore have already been included in \(\Gamma_x\). It cannot be in \(\Gamma_y\), by our definition. Analogous arguments preclude any other intersection among the three sets.

Since \(T\) is a subset of the tetrahedron \(OA_1A_2A_3\), and disjoint from the (disjoint) union \(G_x \cup G_y \cup G_z\), it now follows that

\[
\text{vol}(T) \leq D^3/6 - \text{vol}(G_x) - \text{vol}(G_y) - \text{vol}(G_z).
\]

Notice that

\[
\text{vol}(G_x) + \text{vol}(G_y) + \text{vol}(G_z) = \int \int_{p \in Q} \lambda \delta(p) \, da,
\]

where \(da\) denotes the differential of area, \(\lambda = 1/3\) is a constant introduced because we are integrating over the slanted plane \(P\) instead of over the coordinate planes, and \(\delta(p)\) is the distance from \(p\) to an appropriate coordinate plane. If \(p \in \Gamma_x\) then \(\delta(p)\) is the distance from \(p\) to the \(yz\) plane; if \(p \in \Gamma_y\), then \(\delta(p)\) denotes the distance from \(p\) to the \(xz\) plane, etc.

A lower bound for this double integral over the triangle \(Q\) will thus provide an upper bound for the volume of \(T\). The integral will be smallest when the integral \(\delta(p)\) is as small as possible at every point, and that will be true if \(\delta(p) - \ell_1(p)\), where \(\ell_1(p)\) is the distance from \(p\) to the nearest coordinate plane. Thus,

\[
\text{vol}(T) \leq \frac{D^3}{6} - \int \int_{p \in Q} \lambda \delta_1(p) \, da.
\]

The right side of this inequality can be explicitly integrated, with some difficulty, but it is more easily interpreted as the volume of a star-shaped (Figure 4) object formed by adjoining to the cube \(C - [0, D]^3\) three pyramids slanting from the faces of \(C\) to the points \((D,0,0), (0,D,0),\) and \((0,0,D)\), respectively. Its volume is one-ninth times \(D^3\). Thus, in the case that \(T\) has no notch, we have shown that

\[
\text{vol}(T) \leq \frac{D^3}{9}.
\]
With notch argument. How does the argument change if \( T \) has a notch? Then, not all of the points on \( Q \) are of type \( x, y, \) or \( z \); but those which aren’t must be able to “see” the notch. If \( n \) is the notch, let \( \Delta \) be the set of points \( p \in Q \) such that \( n \leq p \). Following the previous argument, let \( \Gamma_z \) denote the subset of \( Q \setminus \Delta \) comprising all points of type \( x \). Let \( \Gamma_y \) denote the subset of \( Q \setminus \Delta \) comprising all points which are not in \( \Gamma_z \) and which are of type \( y \). Let \( \Gamma_x \) be the set of all points of \( Q \setminus \Delta \) which are not in \( \Gamma_z \cup \Gamma_y \) and which are of type \( z \) (see Figure 5). Then \( Q \) is the disjoint union of \( \Gamma_z, \Gamma_y, \Gamma_x, \) and \( \Delta \).

Again, let \( G_z \) be the union of all segments from \( \Gamma_z \) perpendicular to the \( yz \) plane; let \( G_y \) be the union of all segments from \( \Gamma_y \) perpendicular to the \( xx \) plane; and let \( G_x \) be the union of all segments from \( \Gamma_x \) perpendicular to the \( xy \) plane. The argument of Lemma 3.2 still works, proving that \( G_z, G_y, \) and \( G_x \) are disjoint. Let \( H \) denote the union of all segments from \( \Delta \) to the notch point \( n \). Thus,

\[
\text{vol}(T) \leq D^3/6 - \text{vol}(G_z) - \text{vol}(G_y) - \text{vol}(G_x) - \text{vol}(H).
\]

Now

\[
\text{vol}(G_z) + \text{vol}(G_y) + \text{vol}(G_x) = \int \int_{p \in Q \setminus \Delta} \lambda \delta(p) \, da,
\]
where \( da \) denotes the differential of area, \( \lambda - \frac{1}{\sqrt{3}} \) is a constant introduced because we are integrating over the slanted plane \( P \) instead of over the coordinate planes, and \( \delta(p) \) is the distance from \( p \) to an appropriate coordinate plane (for elements in \( \Gamma_x, \Gamma_y, \) or \( \Gamma_z \)).

In fact, by introducing another constant, \( \lambda' \),

\[
\text{vol}(G_x) + \text{vol}(G_y) + \text{vol}(G_z) - \int_{p \in Q \setminus \Delta} \lambda' \delta'(p) \, da,
\]

where \( \delta'(p) \) is the distance from \( p \) to the intersection of \( Q \) with one of the three coordinate planes (depending on which region, \( \Gamma_x, \Gamma_y, \) or \( \Gamma_z \) \( p \) is in).

Clearly, this integral will be made smaller if the \( \Gamma \) regions are adjusted so that \( \delta'(p) \) is always the distance from \( p \) to the nearest edge of \( Q \) when \( p \notin \Delta \) (see the first diagram in Figure 6).

![Fig. 6. Placement of the delta.](image)

Thus, we may assume that the (new) \( \Gamma \) regions are bounded by the three lines from the vertices of \( Q \) to its center. For convenience, let us refer to those lines as the propeller lines. The only question remaining is where and how big \( \Delta \) should be, in order to minimize the integral

\[
I = \int_{p \in Q \setminus \Delta} \lambda' \delta_1(p) \, da,
\]

where \( \delta_1(p) \) is the distance from \( p \) to the nearest side of \( Q \).

Note that \( Q \) and \( \Delta \) are equilateral triangles with parallel sides. Note also that if none of the vertices of \( \Delta \) is on a propeller line, then at least one edge, \( E, \) of \( \Delta \) lies entirely within the region bounded by the propellers and by the edge of \( Q \) parallel to it (for if each edge crosses a propeller, it meets another edge which is closer to the outside of the triangle, which crosses another propeller and meets... etc.).

**Lemma 3.3.** If we slide \( \Delta, \) keeping its shape, size, and orientation constant, in a direction perpendicular to and away from that edge of \( Q \) which is parallel to \( E, \) the double integral \( I \) will be decreased. (See the first two diagrams of Figure 6.)

**Proof.** The differential for \( I \) is given by the change in that part which is taken over \( Q \setminus \Delta \). Thus,

\[
dI = \left( \int_E \delta_1(p) \, dp \right) \frac{1}{2} \int_b \delta_1(p) \, dp - \frac{1}{2} \int_c \delta_1(p) \, dp + \frac{1}{2} \int_b \delta_1(p) \, dp \right) \, ds,
\]

where \( ds \) is the differential of distance in the direction implied by the lemma statement, and the \( \, dp \) differentials mean that the three single integrals are to be taken over the three sides \( (E, b, \) and \( c) \) with respect to positive distance along those sides. The
factors of \( \frac{1}{2} \) come from the cosine of 60 degrees (since the sides \( b \) and \( c \) are slanted at that angle to \( E \)).

Since the distance from every point on \( b \) (for instance) is at least as far from the nearest edge of \( Q \) as is the entire edge \( E \) (at one end of \( b \) distances are the same, but points near that end are farther away), the integral over \( b \) above is bigger than the one over \( E \). Similarly, the integral over \( c \) is bigger than the one over \( E \). Thus, even with the factors of \( \frac{1}{2} \), the two negative integrals overpower the positive one, and \( dI < 0 \).

**Lemma 3.4.** If one of the vertices of \( \Delta \) is on a propeller line, and if the opposite edge is moved closer to the position where its ends lie on the propeller lines (keeping the one vertex on the propeller line and keeping the size of \( Q \) fixed) the double integral \( I \) will decrease. (This is illustrated by the last two diagrams of Figure 6.)

**Proof.** Letting \( E \) be the edge opposite the vertex \( v \) on a propeller line, either \( E \) is outside of the other two propeller lines, in which case the previous lemma applies, or \( E \) lies on the same side of the center of \( \Delta \) as \( v \) (in which case the assertion is rather trivial), or \( E \) crosses both of the other two propeller lines. Write (as before)

\[
dI = \left( \int_E b_1(p)dp - \frac{1}{2} \int_b b_1(p)|dp| - \frac{1}{2} \int_c b_1(p)|dp| \right) ds,
\]

(where \( b \) and \( c \) are now the two edges emanating from \( v \). Since each of these edges lies entirely in a region bounded by propeller lines, each of the negative integrals in our differential is (strictly) greater than the positive one. Thus a complementary argument (to the proof of the previous lemma) applies. The double integral will be decreased by moving \( v \) closer to the center of \( Q \).)

Now, the problem is reduced to a simple calculus problem. Given that the two triangles have a common center, what size \( \Delta \) maximizes \( D^3/6 - I = \text{vol}(H) \), which describes a specific shape (Figure 7) with volume

\[
u^3 + 3u^2(D - 3u) + \frac{1}{2}u(D - 3u)^2 + \frac{1}{3}u^2(2u),
\]

where \((u, u, u)\) are the coordinates of the notch \( n \), which defines the size of \( \Delta \) on \( Q \) (the smaller \( u \), the larger \( \Delta \) is, but centered on \( Q \))? Taking the derivative and setting it equal to zero gives

\[(5u - D)(3u - D) = 0.\]
The root $u - \frac{D}{3}$ makes the notch vanish and gives the value $\frac{D^2}{3}$ for the volume estimate ($\frac{D^2}{6} - I$). The root $u - \frac{D}{3}$ is a local maximum (i.e., it corresponds to a minimum of our double integral) and gives the value $\frac{3D^2}{20}$ for our volume estimate. Thus, we have the following.

**Theorem 3.5.** If the solid diameter of a three-dimensional Cayley tile $T$ is $D$, then the volume of $T$ is at most $\frac{3D^2}{20}$.

**Corollary 3.6.** If the Cayley graph of a finite Abelian group $G$ with three generators has diameter $d$, then $G$ has at most $\frac{3(d+3)^3}{8}$ elements.

Hsu and Jia [6] claimed to show that $n \leq \frac{(d+3)^3}{8.8}$, which would have been a better bound than the one proved in our paper; however, their proof is flawed. Discussion of the proof of their Theorem 3 is made more difficult by the vagueness of their assumptions. Although they do not state this clearly, it appears that they understood that a tile is uniquely determined by its silhouette and the position of its notch, if it has one. Apparently they believed that in order to maximize the volume of a tile-like shape, all the extreme points of its silhouette must lie along a triangle. However, as we can see in our Figure 7, the silhouette can be more elongated along the axes.

Their paper does, however, contain a beautiful constructive proof that the volume of the largest Cayley tile for a cyclic group with three generators, with a given diameter $d$, is at least $\frac{d^2}{10}$ asymptotically.

4. **Computational results.** The following table contains the largest number of nodes $n$ such that there exists an Abelian group on three generators with diameter $d$ for $d$ up to 17. Table 1 includes the value $\alpha$ such that $n = \alpha(d + 3)^3 = \alpha D^3$. Notice that our bound has value $\alpha = 3/25 - 0.12$ and the best value for actual Cayley graphs in the table below is when $n = 84$, which has $\alpha = 84/(10^3) = 0.084$. The table also compares the solid diameter $D$ to the diameter that a cube of the same volume would have. This is expressed by the number $\beta$ given by $D = \beta(3\sqrt[3]{n})$. In the table (for each diameter $d$ and corresponding largest number of nodes $n$), we give the generators for the first Abelian-Cayley graph found by our computer search. For all cyclic cases, with the exception of $d = 7$, there was an Abelian–Cayley graph of minimal diameter that had 1 as a generator.

The case $d = 17$ shows that the best diameter for $n = 672 - 84 \times 2^3$ was obtained by taking the tile for $n = 84$ and replacing every unit cube by a $2 \times 2 \times 2$ cube. One strategy for possibly improving on the best $\alpha$ and $\beta$ is to take the $n$ which achieves these best values and look at the values of $\alpha$ and $\beta$ obtained from multiples $n \times m^3$, for $m = 2, 3, \ldots$. Since each of these requires a great deal of computer time, only the first several cases, $n = 2268 = 84 \times 3^3$ and $n = 5376 = 84 \times 4^3$, were attempted. Neither case produced better values of $\alpha$ and $\beta$.

5. **Infinite families of tiles.** We can create infinite families of tiles by scaling up existing tiles. Each cube of the smaller tile is replaced by an $m_1$ by $m_2$ by $m_3$ block of cubes. The group of the scaled-up tile will not be cyclic in the case when $\gcd(m_1, m_2, m_3) \neq 1$. The group of the scaled-up tile can be calculated. In fact, it is possible to find the Cayley group and a set of generators of the tile given a translational tiling via Smith Normal Form [5]. If this block is cubical ($m = m_1 - m_2 - m_3$) and the original tile had volume $n$ and diameter $D = d + 3$, then the scaled up version has volume $m^3n$ and diameter $mD$ and hence retains the same values of $\alpha$ and $\beta$. The group of the scaled-up tile will not be cyclic if $m > 1$. The scaled-up version of the tripod shaped tile in the table for $d = 1$ gives a family which is a special case of what
Table 1

<table>
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<th>(n_{\text{max}})</th>
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<th>(\text{Generators})</th>
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<th>(\beta)</th>
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<tr>
<td>2</td>
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<td>1 3 4</td>
<td>0.87200</td>
<td>0.80125</td>
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<tr>
<td>3</td>
<td>16</td>
<td>(\mathbb{Z}_{16})</td>
<td>1 4 5</td>
<td>0.97407</td>
<td>0.79370</td>
</tr>
<tr>
<td>4</td>
<td>27</td>
<td>(\mathbb{Z}_{27})</td>
<td>1 4 17</td>
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<td>0.77778</td>
</tr>
<tr>
<td>5</td>
<td>40</td>
<td>(\mathbb{Z}_{40})</td>
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<td>0.77974</td>
</tr>
<tr>
<td>6</td>
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<td>(\mathbb{Z}_{57})</td>
<td>1 13 33</td>
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<td>0.77952</td>
</tr>
<tr>
<td>7</td>
<td>84</td>
<td>(\mathbb{Z}_{84})</td>
<td>2 9 35</td>
<td>0.98400</td>
<td>0.76112</td>
</tr>
<tr>
<td>8</td>
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<td>0.98340</td>
<td>0.76295</td>
</tr>
<tr>
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<td>0.77405</td>
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<tr>
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<td>0.77325</td>
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<tr>
<td>11</td>
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<td>(\mathbb{Z}_{217})</td>
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<td>0.77658</td>
</tr>
<tr>
<td>12</td>
<td>279</td>
<td>(\mathbb{Z}_{279}\times\mathbb{Z}_3)</td>
<td>(1,0) (1,1) (50,1)</td>
<td>0.98267</td>
<td>0.76519</td>
</tr>
<tr>
<td>13</td>
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<td>0.76414</td>
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<tr>
<td>14</td>
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<td>0.77232</td>
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<tr>
<td>15</td>
<td>462</td>
<td>(\mathbb{Z}_{462})</td>
<td>1 29 97</td>
<td>0.97922</td>
<td>0.77614</td>
</tr>
<tr>
<td>16</td>
<td>560</td>
<td>(\mathbb{Z}_{560})</td>
<td>1 215 326</td>
<td>0.98164</td>
<td>0.76837</td>
</tr>
<tr>
<td>17</td>
<td>672</td>
<td>(\mathbb{Z}_{672}\times\mathbb{Z}_2\times\mathbb{Z}_2)</td>
<td>(2,0,1) (9,0,0) (55,1,0)</td>
<td>0.98400</td>
<td>0.76112</td>
</tr>
</tbody>
</table>

Sherman Stein called semicrystals [8]. One of the nice properties of scaling is that the simplicity of the shape is preserved.

There are also infinite families of Cayley tiles which do not arise from scaling. For example, a family that we call the double staircases has \(n - k^2\) nodes, diameter \(k - 1\), and edge generating list \([1, k, k + 1]\) (see Figure 8 for \(k - 9, n - 81\)). The entry in the table above for \(d - 3\) is a double staircase:

![Double staircase](image-url)

In terms of evaluating the Abelian–Cayley graphs as network topologies, some of these families of tiles have a useful regularity of shape which carries over into physical layout and routing algorithms. Even though the tile in the table above with \(n - 84\) has the lowest diameter as a function of volume, it lacks such regularity. The double staircases and the scaled tripods are two families which manage to combine regularity and low diameter. We have results concerning other families of shapes which will appear in future paper.

6. Open problems. In two dimensions, Wong and Coppersmith [9] showed that a necessary and sufficient condition for a shape to be a tile is that it be L-shaped. In [5], Fiduccia, Zito, and Mann give some conditions on a three-dimensional shape that are necessarily satisfied if the shape is a Cayley tile. However, no set of neces-
sary and sufficient conditions are known for the three-dimensional case. Even less is known about the higher-dimensional cases. What are best diameter tiles in higher dimensions? Can the methods of this paper be generalized to four dimensions and higher? Can the diameter bound in three dimensions be improved?

REFERENCES