Zeros and Maximum Values of Continuous Functions
Computer Assignment 2
Due Thursday, September 17

This assignment introduces properties of the continuous functions that result from properties of the real number system. We will discuss these properties in more depth on Tuesday, September 15.

The Maple worksheet for this assignment is extremely short. The only command it contains is

\[ \text{plot}(\text{BesselJ}(1,x), x=a..b); \]

Replacing \(a\) and \(b\) by two numbers and executing the command will plot a particular continuous function (known as a “Bessel function of the first kind”) on the interval between those two numbers. You will need to execute this command several times in the following exercises.

Recall that a number \(x\) is a zero of a function \(f\) if \(f(x) = 0\). If you’ll have Maple plot this Bessel function on the interval \([3, 5]\), you’ll see that the function appears to have a zero somewhere near 3.8.

1. By choosing a succession of increasingly narrow intervals on which to have Maple plot this Bessel function, see if you can find an interval of width less than 0.000001 on which the function appears to have a zero. List the intervals you tried in the order you tried them.

A function \(f\) has an absolute maximum at a number \(m\) if \(f(m) \geq f(x)\) for every \(x\) in the domain of \(f\). If you’ll have Maple plot this Bessel function on the interval \([-20, 20]\), you’ll see that it seems to have an absolute maximum somewhere near 2, and the value of the function at that point looks like it’s around 0.6.

2. By choosing a succession of increasingly narrow intervals on which to have Maple plot this Bessel function, see if you can find an interval of width less than 0.000001 on which the function appears to have an absolute maximum. List the intervals you tried in the order you tried them. Estimate the absolute maximum value of the function (i.e., the value of the function at the point where the function has an absolute maximum).
As we’ll see when we discuss Section 2.6 in the text, there are two important theorems that justify what you did in the two exercises above. In particular, a theorem called the Intermediate Value Theorem guarantees that if you have a function \( f \) that is continuous on an interval \([a, b]\), and \( f(a) \) and \( f(b) \) have different signs, then there must be at least one zero of \( f \) in between \( a \) and \( b \). Also, a theorem called the Extreme Value Theorem guarantees that if you have a continuous function \( f \) whose domain is an interval \([a, b]\), then \( f \) has an absolute maximum somewhere on that interval.

Both of these theorems depend on the fact that the set of real numbers is defined in such a way that it has no “holes”. Other number systems may lack this property, and these two theorems fail if we restrict ourselves to such systems. For example, if we restrict ourselves to only using rational numbers, as the ancient Greeks did, the function \( f(x) = x^2 - 2 \) has no zero on the interval \([1, 2]\) even though \( f(1) < 0 \) and \( f(2) > 0 \). If there were a zero on this interval, it would be a square root of 2, and there is no rational number whose square is 2.

3. Prove that there is no rational number whose square is 2, by completing the following steps, which outline what is known as a proof by contradiction. We’ll suppose that there actually is a rational number whose square is 2, and find out that this leads to impossible conclusions. This means that the supposition must have been wrong.

(a) Explain why, if \( \sqrt{2} \) is rational, we can write \( \sqrt{2} \) as the quotient \( p/q \) of two integers, in such a way that \( p \) and \( q \) are not both even. (Note that saying that \( p \) and \( q \) are not both even is not the same as saying that they are both not even.)

(b) Note that if \( p/q = \sqrt{2} \) then

\[
p^2 = 2q^2. \tag{\ast}
\]

Explain why, if \( p \) is odd, the left side of (\ast) would be odd but the right side of (\ast) would be even. Could (\ast) be true in that case?

(c) If \( p \) and \( q \) aren’t both even, and \( p \) isn’t odd, that must mean that \( p \) is even and \( q \) is odd. In this case, would the left side of (\ast) be divisible by 4? Would the right side of (\ast) be divisible by 4? Could (\ast), then, be true in this case?