Rearrangements

Definition. A series $\sum b_k$ is a rearrangement of a series $\sum a_k$ if terms in the two series are the same but they appear in a (possibly) different order. (More precisely, there is an invertible function $f : \mathbb{N} \to \mathbb{N}$ such that $b_k = a_{f(k)}$ for every $k \in \mathbb{N}$.)

Consider, for example, the alternating harmonic series:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \cdots.$$ 

One rearrangement of it is the following:

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots.$$ 

The following is not a rearrangement of the alternating harmonic series:

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} - \cdots.$$ 

Here, the attempt is to list all of the positive terms first, followed by all of the negative terms. This is not even a series. (Which term is $-1/4$? The “$(\infty + 1)$th”?)

Theorem. Given any conditionally convergent series $\sum a_k$ and any number $s$, there is a rearrangement of $\sum a_k$ that converges to $s$.

Notice that if you take any $n$ terms in the alternating harmonic series and add them together, this will be the $n$th partial sum of some rearrangement of the alternating harmonic series.

Exercise 1 Let $L$ be the number you get when you take your student identification number (which is probably your social security number) and place a decimal point in front of it, and let $\varepsilon = 0.001$. By experimenting with the computer, find a partial sum, $s_n$, of a rearrangement of the alternating harmonic series such that $|L - s_n| < \varepsilon$. (You should tell me what terms you used in this partial sum, but you don’t need to tell me what rearrangement of the alternating harmonic series this is a partial sum of.)
Fact. Suppose you pick any $N$ terms from the alternating harmonic series in any order and label them $c_1, c_2, \ldots, c_N$. Let $y = c_1 + c_2 + \cdots + c_N$ and let $\delta$ be the size of the largest term in the alternating harmonic series that you did not pick. Then there is a rearrangement $\sum b_k$ of the alternating harmonic series such that:

- $b_k = c_k$ for $k = 1, 2, \ldots, N$;
- $\left| \sum_{k=1}^{n} b_k - y \right| \leq \delta$ for every $n \geq N$.

In other words, your $N$-term sum can be extended to a rearrangement of the alternating harmonic series in such a way that the $n$th partial sum of the rearrangement is within $\delta$ of the $N$th partial sum of the rearrangement for every $n \geq N$.

This fact can be verified using a strategy similar to one discussed in class.

**Exercise 2** Let $L$ be the same number as in Exercise 1 and let $\varepsilon = 0.01$. By experimenting with the computer and using the fact mentioned above, find $N$ terms $c_1, c_2, \ldots, c_N$ of the alternating harmonic series that can be extended to a rearrangement, $\sum c_n$, of the alternating harmonic series such that $\left| L - \sum_{k=1}^{n} c_k \right| < \varepsilon$ for every $n \geq N$. (You should tell me what the terms $c_1, \ldots, c_N$ you chose are, but you don’t need to tell me what the entire rearrangement of the alternating harmonic series, nor do you need to prove that the terms you chose work.)

Notice that Exercises 1 and 2 have different requirements. In Exercise 2, $\varepsilon$ is larger than it was in Exercise 1, but in Exercise 2 all partial sums past a certain point must be within $\varepsilon$ of $L$. 
Approximating $\pi$

Here are the sums of some series:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6},$$
$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90},$$
$$\sum_{k=1}^{\infty} \frac{1}{k^6} = \frac{\pi^6}{945},$$
$$\sum_{k=1}^{\infty} \frac{1}{k^8} = \frac{\pi^8}{9450}.$$

We can use partial sums of these series to approximate $\pi$. For example, the first equation implies that

$$\pi \approx \left(6 \sum_{k=1}^{n} \frac{1}{k^2}\right)^{1/2}$$

if $n$ is large. (The other equations can be similarly solved for $\pi$.)

**Exercise 3**

(a) How many terms are in the shortest partial sum of $\sum (1/k^2)$ that can be used to approximate $\pi$ within 0.0001?

(b) How many terms are in the shortest partial sum of $\sum (1/k^4)$ that can be used to approximate $\pi$ within 0.0001?

(c) How many terms are in the shortest partial sum of $\sum (1/k^6)$ that can be used to approximate $\pi$ within 0.0001?

(d) How many terms are in the shortest partial sum of $\sum (1/k^8)$ that can be used to approximate $\pi$ within 0.0001?