Lyapunov's Direct Method

Lecture 22
Math 634
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An other tool for determining stability of solutions is *Lyapunov's direct method*. While this method may actually seem rather indirect, it does work directly on the equation in question instead of on its linearization.

We will consider this method for equilibrium solutions of (possibly) nonautonomous equations. Let $\Omega \subseteq \mathbb{R}^n$ be open and contain the origin, and suppose that $f : \mathbb{R} \times \Omega \to \mathbb{R}^n$ is a continuously differentiable function. Suppose, furthermore, that $f(t, 0) = 0$ for every $t \in \mathbb{R}$, so $x(t) := 0$ is a solution of the equation

$$\dot{x} = f(t, x). \tag{1}$$

(The results we obtain in this narrow context can be applied to determine the stability of other constant solutions of (1) by translation.)

In this lecture, a subset of Ω that contains the origin in its interior will be called a *neighborhood* of 0.

Definition Suppose that $\mathcal{D}$ is a neighborhood of 0 and that $W : \mathcal{D} \to \mathbb{R}$ is continuous and satisfies $W(0) = 0$. Then:

- If $W(x) \geq 0$ for every $x \in \mathcal{D}$, then W is *positive semidefinite*.
- If $W(x) > 0$ for every $x \in \mathcal{D} \setminus \{0\}$, then W is *positive definite*.
- If $W(x) \leq 0$ for every $x \in \mathcal{D}$, then W is *negative semidefinite*.
- If $W(x) < 0$ for every $x \in \mathcal{D} \setminus \{0\}$, then W is *negative definite*.

Definition Suppose that $\mathcal{D}$ is a neighborhood of 0 and that $V : \mathbb{R} \times \mathcal{D} \to \mathbb{R}$ is continuous and satisfies $V(t, 0) = 0$ for every $t \in \mathbb{R}$. Then:

- If there is a positive semidefinite function $W : \mathcal{D} \to \mathbb{R}$ such that $V(t, x) \geq W(x)$ for every $(t, x) \in \mathbb{R} \times \mathcal{D}$, then V is *positive semidefinite*.
- If there is a positive definite function $W : \mathcal{D} \to \mathbb{R}$ such that $V(t, x) \geq W(x)$ for every $(t, x) \in \mathbb{R} \times \mathcal{D}$, then V is *positive definite*.

- If there is a negative semidefinite function $W : \mathcal{D} \to \mathbb{R}$ such that $V(t, x) \leq W(x)$ for every $(t, x) \in \mathbb{R} \times \mathcal{D}$, then V is *negative semidefinite.*
- If there is a negative definite function $W : \mathcal{D} \to \mathbb{R}$ such that $V(t, x) \leq W(x)$ for every $(t, x) \in \mathbb{R} \times \mathcal{D}$, then V is *negative definite.*

Definition If $V : \mathbb{R} \times \mathcal{D}$ is continuously differentiable then its *orbital derivative* (w.r.t. (1)) is the function $\dot{V} : \mathbb{R} \times \mathcal{D} \to \mathbb{R}$ given by the formula

$$\dot{V}(t, x) := \frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) \cdot f(t, x).$$

(Here "$\partial V(t, x)/\partial x$" represents the gradient of the function $V(t, \cdot)$.)

Note that if $x(t)$ is a solution of (1) then, by the chain rule,

$$\frac{d}{dt}V(t, x(t)) = \dot{V}(t, x(t)).$$

A function whose orbital derivative is always nonpositive is sometimes called a *Lyapunov function.*

Theorem (Lyapunov Stability) *If there is a neighborhood $\mathcal{D}$ of 0 and a continuously differentiable positive definite function $V : \mathbb{R} \times \mathcal{D} \to \mathbb{R}$ whose orbital derivative $\dot{V}$ is negative semidefinite, then 0 is a Lyapunov stable solution of (1).*

Proof. Let $\varepsilon > 0$ and $t_0 \in \mathbb{R}$ be given. Assume, without loss of generality, that $\overline{B(0, \varepsilon)}$ is contained in $\mathcal{D}$. Pick a positive definite function $W : \mathcal{D} \to \mathbb{R}$ such that $V(t, x) \geq W(x)$ for every $(t, x) \in \mathbb{R} \times \mathcal{D}$. Let

$$m := \min\{W(x) \mid |x| = \varepsilon\}.$$

Since W is continuous and positive definite, m is well-defined and positive. Pick $\delta > 0$ small enough that $\delta < \varepsilon$ and

$$\max\{V(t_0, x) \mid |x| \leq \delta\} < m.$$

(Since V is positive definite and continuous, this is possible.)

Now, if $x(t)$ solves (1) and $|x(t_0)| < \delta$ then $V(t_0, x(t_0)) < m$, and

$$\frac{d}{dt}V(t, x(t)) = \dot{V}(t, x(t)) \leq 0,$$

for all t, so $V(t,x(t)) < m$ for every $t \geq t_0$. Thus, $W(x(t)) < m$ for every $t \geq t_0$, so, for every $t \geq t_0$, $|x(t)| \neq \varepsilon$. Since $|x(t_0)| < \varepsilon$, this tells us that $|x(t)| < \varepsilon$ for every $t \geq t_0$. □

Theorem (Asymptotic Stability) *Suppose that there is a neighborhood $\mathcal{D}$ of 0 and a continuously differentiable positive definite function $V : \mathbb{R} \times \mathcal{D} \to \mathbb{R}$ whose orbital derivative $\dot{V}$ is negative definite, and suppose that there is a positive definite function $\overline{W} : \mathcal{D} \to \mathbb{R}$ such that $V(t,x) \leq \overline{W}(x)$ for every $(t,x) \in \mathbb{R} \times \mathcal{D}$. Then 0 is an asymptotically stable solution of (1).*

Proof. By the previous theorem, 0 is a Lyapunov stable solution of (1). Let $t_0 \in \mathbb{R}$ be given. Assume, without loss of generality, that $\mathcal{D}$ is compact. By Lyapunov stability, we know that we can choose a neighborhood $\mathcal{U}$ of 0 such that if $x(t)$ is a solution of (1) and $x(t_0) \in \mathcal{U}$, then $x(t) \in \mathcal{D}$ for every $t \geq t_0$. We claim that, in fact, if $x(t)$ is a solution of (1) and $x(t_0) \in \mathcal{U}$, then $x(t) \to 0$ as $t \uparrow \infty$. Verifying this claim will prove the theorem.

Suppose that $V(t,x(t))$ does not converge to 0 as $t \uparrow \infty$. The negative definiteness of $\dot{V}$ implies that $V(\cdot, x(\cdot))$ is nonincreasing, so, since $V \geq 0$, there must be a number $c > 0$ such that $V(t,x(t)) \geq c$ for every $t \geq t_0$. Then $\overline{W}(x(t)) \geq c > 0$ for every $t \geq t_0$. Since $\overline{W}(0) = 0$ and $\overline{W}$ is continuous,

$$\inf\{|x(t)| \mid t \geq t_0\} \geq \varepsilon \tag{2}$$

for some constant $\varepsilon > 0$. Pick a negative definite function $Y : \mathcal{D} \to \mathbb{R}$ such that $\dot{V}(t,x) \leq Y(x)$ for every $(t,x) \in \mathbb{R} \times \mathcal{D}$. The compactness of $\mathcal{D} \setminus \overline{B(0,\varepsilon)}$, along with (2), implies that

$$\{Y(x(t)) \mid t \geq t_0\}$$

is bounded away from 0. This, in turn, implies that

$$\{\dot{V}(t,x(t)) \mid t \geq t_0\}$$

is bounded away from 0. In other words,

$$\frac{d}{dt}V(t,x(t)) = \dot{V}(t,x(t)) \leq -\delta \tag{3}$$

for some constant $\delta > 0$. Clearly, (3) contradicts the nonnegativity of V for large t.

That contradiction implies that $V(t, x(t)) \to 0$ as $t \uparrow \infty$. Pick a positive definite function $\underline{W} : \mathcal{D} \to \mathbb{R}$ such that $V(t, x) \geq \underline{W}(x)$ for every $(t, x) \in \mathbb{R} \times \mathcal{D}$, and note that $\underline{W}(x(t)) \to 0$ as $t \uparrow \infty$.

Let $r > 0$ be given, and let

$$w_r = \min\{\underline{W}(p) \mid p \in \mathcal{D} \setminus \overline{B(0, r)}\},$$

which is defined and positive by the compactness of $\mathcal{D}$ and the continuity and positive definiteness of $\underline{W}$. Since $\underline{W}(x(t)) \to 0$ as $t \uparrow \infty$, there exists T such that $\underline{W}(x(t)) < w_r$ for every $t > T$. Thus, for $t > T$, it must be the case that $x(t) \in \overline{B(0, r)}$. Hence, 0 is asymptotically stable. □

It may seem strange that we ned to bound V by a time-independent, positive definite function $\overline{W}$ from above. Indeed, some textbooks (see, *e.g.*, Theorem 2.20 in *Stability, Instability, and Chaos* by Glendinning) contain asymptotic stability theorems omitting this hypothesis. A counterexample by Massera demonstrates the necessity of the hypothesis.

<u>Exercise 14</u> Show, by means of a counterexample, that the theorem on asymptotic stability via Lyapunov's direct method fails if the hypothesis about $\overline{W}$ is dropped.
(You may, but do not have to, proceed as follows. Let $g : \mathbb{R} \to \mathbb{R}$ be a function that is twice continuously differentiable and satisfies $g(t) \geq e^{-t}$ for every $t \in \mathbb{R}$, $g(t) \leq 1$ for every $t \geq 0$, $g(t) = e^{-t}$ for every

$$t \notin \bigcup_{n \in \mathbb{N}} (n - 2^{-n}, n + 2^{-n}),$$

and $g(n) = 1$ for every $n \in \mathbb{N}$. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the function defined by the formula

$$f(t, x) := \frac{g'(t)}{g(t)} x,$$

and let $V : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the function defined by the formula

$$V(t, x) := \frac{x^2}{[g(t)]^2} \left[3 - \int_0^t [g(\tau)]^2 \, d\tau \right].$$

Show that, for x near 0, $V(t, x)$ is positive definite, $\dot{V}(t, x)$ is negative definite, and the solution 0 of (1) is not asymptotically stable.)