Poincaré-Bendixson Theorem

Lecture 39
Math 634
12/3/99

Definition A periodic orbit of a continuous dynamical system \( \varphi \) is a set of the form

\[
\{ \varphi(t, p) \mid t \in [0, T] \}
\]

for some time \( T \) and point \( p \) satisfying \( \varphi(T, p) = p \). If this set is a singleton, we say that the periodic orbit is degenerate.

Theorem (Poincaré-Bendixson) Every nonempty, compact \( \omega \)-limit set of a \( C^1 \) planar flow that does not contain an equilibrium point is a (nondegenerate) periodic orbit.

We will prove this theorem by means of 4 lemmas. Throughout our discussion, we will be referring to a \( C^1 \) planar flow \( \varphi \) and the corresponding vector field \( f \).

Definition If \( S \) is a line segment in \( \mathbb{R}^2 \) and \( p_1, p_2, \ldots \) is a (possibly finite) sequence of points lying on \( S \), then we say that this sequence is monotone on \( S \) if \( (p_j - p_{j-1}) \cdot (p_2 - p_1) \geq 0 \) for every \( j \geq 2 \).

Definition A (possibly finite) sequence \( p_1, p_2, \ldots \) of points on a trajectory \( T \) of \( \varphi \) is said to be monotone on \( T \) if we can choose a point \( p \) and times \( t_1 \leq t_2 \leq \cdots \) such that \( \varphi(t_j, p) = p_j \) for each \( j \).

Definition A transversal of \( \varphi \) is a line segment \( S \) such that \( f \) is not tangent to \( S \) at any point of \( S \).

Lemma If a (possibly finite) sequence of points \( p_1, p_2, \ldots \) lies on the intersection of a transversal \( S \) and a trajectory \( T \), and the sequence is monotone on \( T \), then it is monotone on \( S \).

Proof. Let \( p \) be a point on \( T \). Since \( S \) is closed and \( f \) is nowhere tangent to \( S \), the times \( t \) at which \( \varphi(t, p) \in S \) form an increasing sequence (possibly
biinfinite). Consequently, if the lemma fails then there are times $t_1 < t_2 < t_3$ and distinct points $p_i = \varphi(t_i, p) \in \mathcal{S}$, $i \in \{1, 2, 3\}$, such that

$$\{p_1, p_2, p_3\} = \varphi([t_1, t_3], p) \cap \mathcal{S}$$

and $p_3$ is between $p_1$ and $p_2$. Note that the union of the line segment $p_1p_2$ from $p_1$ to $p_2$ with the curve $\varphi([t_1, t_2], p)$ is a simple closed curve in the plane, so by the Jordan Curve Theorem it has an “inside” $\mathcal{I}$ and an “outside” $\mathcal{O}$. Assuming, without loss of generality, that $f$ points into $\mathcal{I}$ all along the “interior” of $p_1p_2$, we get a picture something like:

![Diagram showing the flow box](image)

Note that

$$\mathcal{I} \cup \overline{p_1p_2} \cup \varphi([t_1, t_2], p)$$

is a positively invariant set, so, in particular, it contains $\varphi([t_2, t_3], p)$. But the fact that $p_3$ is between $p_1$ and $p_2$ implies that $f(p_3)$ points into $\mathcal{I}$, so $\varphi(t_3 - \varepsilon, p) \in \mathcal{O}$ for $\varepsilon$ small and positive. This contradiction implies that the lemma holds.

The proof of the next lemma uses something called a flow box. A flow box is a (topological) box such that $f$ points into the box along one side, points out of the box along the opposite side, and is tangent to the other
two sides, and the restriction of $\varphi$ to the box is conjugate to unidirectional, constant-velocity flow. The existence of a flow box around any regular point of $\varphi$ is a consequence of the $C^r$-rectification Theorem.

**Lemma** No $\omega$-limit set intersects a transversal in more than one point.

*Proof.* Suppose that for some point $x$ and some transversal $S$, $\omega(x)$ intersects $S$ at two distinct points $p_1$ and $p_2$. Since $p_1$ and $p_2$ are on a transversal, they are regular points, so we can choose disjoint subintervals $S_1$ and $S_2$ of $S$ containing, respectively, $p_1$ and $p_2$, and, for some $\varepsilon > 0$, define flow boxes $B_1$ and $B_2$ by

$$B_i := \{ \varphi(t, x) \mid t \in [-\varepsilon, \varepsilon], x \in S_i \}.$$

Now, the fact that $p_1, p_2 \in \omega(x)$ means that we can pick an increasing sequence of times $t_1, t_2, \ldots$ such that $\varphi(t_j, x) \in B_1$ if $j$ is odd and $\varphi(t_j, x) \in B_2$ if $j$ is even. In fact, because of the nature of the flow in $B_1$ and $B_2$, we can assume that $\varphi(t_j, x) \in S$ for each $j$. Although the sequence $\varphi(t_1, x), \varphi(t_2, x), \ldots$ is monotone on the trajectory $T := \gamma(x)$, it is not monotone on $S$, contradicting the previous lemma. \hfill \Box

**Definition** An $\omega$-limit point of a point $p$ is an element of $\omega(p)$.

**Lemma** Every $\omega$-limit point of an $\omega$-limit point lies on a periodic orbit.

*Proof.* Suppose that $p \in \omega(q)$ and $q \in \omega(r)$. If $p$ is a singular point, then it obviously lies on a (degenerate) periodic orbit, so suppose that $p$ is a regular point. Pick $S$ to be a transversal containing $p$ in its “interior”. By putting a suitable flow box around $p$, we see that, since $p \in \omega(q)$, the solution beginning at $q$ must repeatedly cross $S$. But $q \in \omega(r)$ and $\omega$-limit sets are invariant, so the solution beginning at $q$ remains confined within $\omega(r)$. Since $\omega(r) \cap S$ contains at most one point, the solution beginning at $q$ must repeatedly cross $S$ at the same point; i.e., $q$ lies on a periodic orbit. Since $p \in \omega(q)$, $p$ must lie on this same periodic orbit. \hfill \Box

**Lemma** If an $\omega$-limit set $\omega(x)$ contains a nondegenerate periodic orbit $\mathcal{P}$, then $\omega(x) = \mathcal{P}$.
Proof. Fix $q \in \mathcal{P}$. Pick $T > 0$ such that $\varphi(T, q) = q$. Let $\varepsilon > 0$ be given. By continuous dependence, we can pick $\delta > 0$ such that $|\varphi(t, y) - \varphi(t, q)| < \varepsilon$ whenever $t \in [0, 3T/2]$ and $|y - q| < \delta$. Pick a transversal $\mathcal{S}$ of length less than $\delta$ with $q$ in its “interior”, and create a flow box

$$B := \{ \varphi(t, x) \mid x \in \mathcal{S}, t \in [-\rho, \rho] \}$$

for some $\rho \in (0, T/4]$. By continuity of $\varphi(T, \cdot)$, we know that we can pick a subinterval $\mathcal{S}'$ of $\mathcal{S}$ that contains $q$ and that satisfies $\varphi(T, \mathcal{S}') \subset B$. Let $t_j$ be the $j$th smallest element of

$$\{ t \geq 0 \mid \varphi(t, x) \in \mathcal{S}' \}.$$

Because $\mathcal{S}'$ is a transversal and $q \in \omega(x)$, the $t_j$ are well-defined and increase to infinity as $j \uparrow \infty$. Also, by the lemma on monotonicity, $|\varphi(t_j, x) - q|$ is a decreasing function of $j$.

Note that for each $j \in \mathbb{N}$, $\varphi(T, \varphi(t_j, x)) \in B$, so, by construction of $\mathcal{S}$ and $B$, $\varphi(t, \varphi(T, \varphi(t_j, x))) \in \mathcal{S}$ for some $t \in [-T/2, T/2]$. Pick such a $t$. The lemma on monotonicity implies that

$$\varphi(t, \varphi(T, \varphi(t_j, x))) \in \mathcal{S}'.$$

This, in turn, implies that $t + T + t_j \in \{ t_1, t_2, \ldots \}$, so

$$t_{j+1} - t_j \leq 3T/2. \quad (1)$$

Now, suppose that $t \geq t_1$. Then $t \in [t_j, t_{j+1})$ for some $j \geq 1$. For this $j$,

$$|\varphi(t, x) - \varphi(t - t_j, p)| = |\varphi(t - t_j, \varphi(t_j, x)) - \varphi(t - t_j, p)| < \varepsilon,$$

since, by (1), $|t - t_j| < |t_{j+1} - t_j| < 3T/2$ and since, because $\varphi(t_j, x) \in \mathcal{S}' \subseteq \mathcal{S}$, $|p - \varphi(t_j, x)| < \delta$.

Since $\varepsilon$ was arbitrary, we have shown that

$$\lim_{t \uparrow \infty} d(\varphi(t, x), \mathcal{P}) = 0.$$

Thus, $\mathcal{P} = \omega(x)$, as was claimed. \qed

Now, we get to the proof of the Poincaré-Bendixson Theorem itself. Suppose $\omega(x)$ is compact and nonempty. Pick $p \in \omega(x)$. Since $\gamma^+(p)$ is contained in the compact set $\omega(x)$, we know $\omega(p)$ is nonempty, so we can pick $q \in \omega(p)$. Note that $q$ is an $\omega$-limit point of an $\omega$-limit point, so, by the third lemma, $q$ lies on a periodic orbit $\mathcal{P}$. Since $\omega(p)$ is invariant, $\mathcal{P} \subseteq \omega(p) \subseteq \omega(x)$. If $\omega(x)$ contains no equilibrium point, then $\mathcal{P}$ is nondegenerate, so, by the fourth lemma, $\omega(x) = \mathcal{P}$.  

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