Picard-Lindelöf Theorem
Lecture 4
Math 634
9/8/99

Theorem The space $\mathcal{C}([a,b])$ of continuous functions from $[a,b]$ to $\mathbb{R}^n$ equipped with the norm

$$
\|f\|_\infty := \sup\{ |f(x)| \mid x \in [a,b] \}
$$

is a Banach space.

Definition Two different norms $\| \cdot \|_1$ and $\| \cdot \|_2$ on a vector space $\mathcal{X}$ are equivalent if there exist constants $m, M > 0$ such that

$$
m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1
$$

for every $x \in \mathcal{X}$.

Theorem If $(\mathcal{X}, \| \cdot \|_1)$ is a Banach space and $\| \cdot \|_2$ is equivalent to $\| \cdot \|_1$ on $\mathcal{X}$, then $(\mathcal{X}, \| \cdot \|_2)$ is a Banach space.

Theorem A closed subspace of a complete metric space is a complete metric space.

We are now in a position to state and prove the Picard-Lindelöf Existence-Uniqueness Theorem. Recall that we are dealing with the IVP

$$
\begin{align*}
\dot{x} &= f(t, x) \\
x(t_0) &= a.
\end{align*}
$$

(1)

Theorem (Picard-Lindelöf) Suppose $f : [t_0 - \alpha, t_0 + \alpha] \times \overline{B(a, \beta)} \to \mathbb{R}^n$ is continuous and bounded by $M$. Suppose, furthermore, that $f(t, \cdot)$ is Lipschitz continuous with Lipschitz constant $L$ for every $t \in [t_0 - \alpha, t_0 + \alpha]$. Then (1) has a unique solution defined on $[t_0 - b, t_0 + b]$, where $b = \min\{\alpha, \beta/M\}$.

Proof. Let $\mathcal{X}$ be the set of continuous functions from $[t_0 - b, t_0 + b]$ to $\overline{B(a, \beta)}$. The norm

$$
\|g\|_w := \sup\{ e^{-2L|t-t_0|} |g(t)| \mid t \in [t_0 - b, t_0 + b] \}
$$

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is equivalent to the standard supremum norm \( \| \cdot \|_\infty \) on \( C([t_0 - b, t_0 + b]) \), so this vector space is complete under this weighted norm. The set \( \mathcal{X} \) endowed with this norm/metric is a closed subset of this complete Banach space, so \( \mathcal{X} \) equipped with the metric \( d(x_1, x_2) := \| x_1 - x_2 \|_w \) is a complete metric space.

Given \( x \in \mathcal{X} \), define \( T(x) \) to be the function on \([t_0 - b, t_0 + b]\) given by the formula

\[
T(x)(t) = a + \int_{t_0}^t f(s, x(s)) \, ds.
\]

**Step 1:** If \( x \in \mathcal{X} \) then \( T(x) \) makes sense. This should be obvious.

**Step 2:** If \( x \in \mathcal{X} \) then \( T(x) \in \mathcal{X} \).

If \( x \in \mathcal{X} \), then it is clear that \( T(x) \) is continuous (and, in fact, differentiable). Furthermore, for \( t \in [t_0 - b, t_0 + b] \)

\[
|T(x)(t) - a| = \left| \int_{t_0}^t f(s, x(s)) \, ds \right| \leq \int_{t_0}^t |f(s, x(s))| \, ds \leq Mb \leq \beta,
\]

so \( T(x)(t) \in \overline{B(a, \beta)} \). Hence, \( T(x) \in \mathcal{X} \).

**Step 3:** \( T \) is a contraction on \( \mathcal{X} \).

Let \( x, y \in \mathcal{X} \), and note that \( \| T(x) - T(y) \|_w \) is

\[
\sup \left\{ e^{-2L|t-t_0|} \left( \int_{t_0}^t \left| f(s, x(s)) - f(s, y(s)) \right| \, ds \right) \bigg| \quad t \in [t_0 - b, t_0 + b] \right\}.
\]

For a fixed \( t \in [t_0 - b, t_0 + b] \),

\[
e^{-2L|t-t_0|} \int_{t_0}^t \left| f(s, x(s)) - f(s, y(s)) \right| \, ds \\
\leq e^{-2L|t-t_0|} \int_{t_0}^t \left| f(s, x(s)) - f(s, y(s)) \right| \, ds \\
\leq e^{-2L|t-t_0|} \int_{t_0}^t L|x(s) - y(s)| \, ds \\
\leq Le^{-2L|t-t_0|} \int_{t_0}^t \| x - y \|_w e^{2L|s-t_0|} \, ds \\
= \frac{\| x - y \|_w}{2} \left( 1 - e^{-2L|t-t_0|} \right) \\
\leq \frac{1}{2} \| x - y \|_w.
\]
Taking the supremum over all $t \in [t_0 - b, t_0 + b]$, we find that $T$ is a contraction (with $\lambda = 1/2$).

By the contraction mapping principle, we therefore know that $T$ has a unique fixed point in $X$. This means that (1) has a unique solution in $X$ (which is the only place a solution could be). \qed