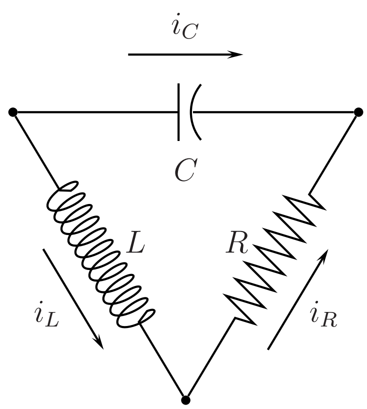
Lienard's Equation

Lecture 40

Math 634

12/6/99

Suppose we have a simple electrical circuit with a resistor, an inductor, and a capacitor as shown.



Kirchhoff's current law tells us that

$$i_L = i_R = -i_C, \tag{1}$$

and Kirchhoff's voltage law tells us that the corresponding voltage drops satisfy

$$V_C = V_L + V_R. \tag{2}$$

By definition of the capacitance C,

$$C\frac{dV_C}{dt} = i_C, \tag{3}$$

and by Faraday's Law

$$L\frac{di_L}{dt} = V_L, \tag{4}$$

where L is the inductance of the inductor. We assume that the resistor behaves nonlinearly and satisfies the generalized form of Ohm's Law:

$$V_R = F(i_R). \tag{5}$$

Let $x = i_L$ and $f(u) := F'(u)$. By (4),

$$\dot{x} = \frac{1}{L} V_L,$$

so by (2), (3), (5), and (1)

$$\begin{aligned} \ddot{x} &= \frac{1}{L}\frac{dV_L}{dt} = \frac{1}{L}(\dot{V}_C - \dot{V}_R) = \frac{1}{L}\left(\frac{1}{C} i_C - F'(i_R)\frac{di_R}{dt}\right) \\ &= \frac{1}{L}\left(\frac{1}{C}(-x) - f(x)\dot{x}\right) \end{aligned}$$

Hence,

$$\ddot{x} + \frac{1}{L} f(x)\dot{x} + \frac{1}{LC} x = 0.$$

By rescaling f and t (or, equivalently, by choosing units judiciously), we get *Lienard's Equation*:

$$\ddot{x} + f(x)\dot{x} + x = 0.$$

We will study Lienard's Equation under the following assumptions on F and f:

(i) $F(0) = 0$;

(ii) f is Lipschitz continuous;

(iii) F is odd;

(iv) $F(x) \to \infty$ as $x \uparrow \infty$;

(v) for some $\beta > 0$, $F(\beta) = 0$ and F is positive and increasing on (β, ∞);

(vi) for some $\alpha > 0$, $F(\alpha) = 0$ and F is negative on $(0, \alpha)$.

Assumption **(vi)** corresponds to the existence of a region of negative resistance. Apparently, there are semiconductors called "tunnel diodes" that behave this way.

By setting $y = \dot{x} + F(x)$, we can rewrite Lienard's Equation as the first-order system

$$\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -x. \end{cases} \tag{6}$$

Definition A *limit cycle* for a flow is a nondegenerate periodic orbit $\mathcal{P}$ that is the ω-limit set or the α-limit set of some point $q \notin \mathcal{P}$.

Theorem (Lienard's Theorem) *The flow generated by* (6) *has at least one limit cycle. If $\alpha = \beta$ then this limit cycle is the only nondegenerate periodic orbit, and it is the ω-limit set of all points other than the origin.*

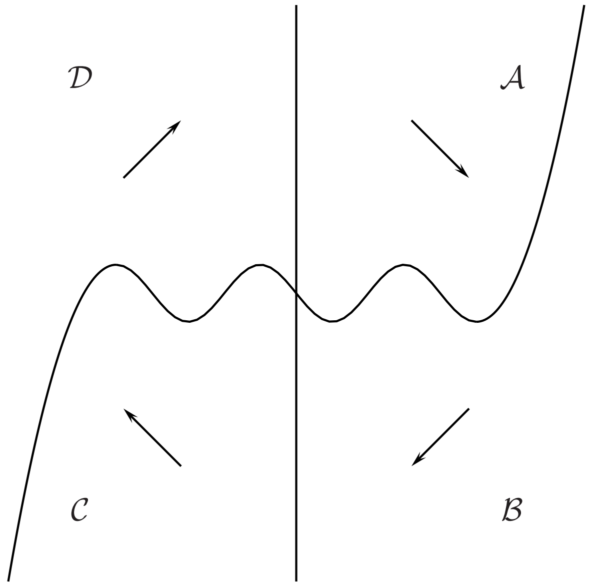
The significance of Lienard's Theorem can be seen by comparing Lienard's Equation with the linear equation that would have resulted if we had assumed a linear resistor. Such linear RCL circuits can have oscillations with arbitrary amplitude. Lienard's Theorem says that, under suitable hypotheses, a nonlinear resistor selects oscillations of one particular amplitude.

We will prove the first half of Lienard's Theorem by finding a compact, positively invariant region that does not contain an equilibrium point and then using the Poincaré-Bendixson Theorem. Note that the origin is the only equilibrium point of (6). Since

$$\frac{d}{dt}(x^2 + y^2) = 2(x\dot{x} + y\dot{y}) = -2xF(x),$$

assumption **(vi)** implies that for ε small, $\mathbb{R}^2 \setminus \mathcal{B}(0, \varepsilon)$ is positively invariant.

The nullclines $x = 0$ and $y = F(x)$ of (6) separate the plane into four regions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$, and the general direction of flow in those regions is as show below. Note that away from the origin, the speed of trajectories is bounded below, so every solution of (6) (except $(x, y) = (0, 0)$ passes through $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ in succession an infinite number of times as it circles around the origin in a clockwise direction.



We claim that if a solution starts at a point $(0, y_0)$ that is high enough up on the positive y-axis, then the first point $(0, \tilde{y}_0)$ it hits on the negative y-axis is closer to the origin then $(0, y_0)$ was. Assume, for the moment, that this claim is true. Let $\mathcal{S}_1$ be the orbit segment connecting $(0, y_0)$ to $(0, \tilde{y}_0)$. Because of the symmetry in (6), the set

$$\mathcal{S}_2 := \big\{(x, y) \in \mathbb{R}^2 \;\big|\; (-x, -y) \in \mathcal{S}_1\big\}$$

is also an orbit segment. Let

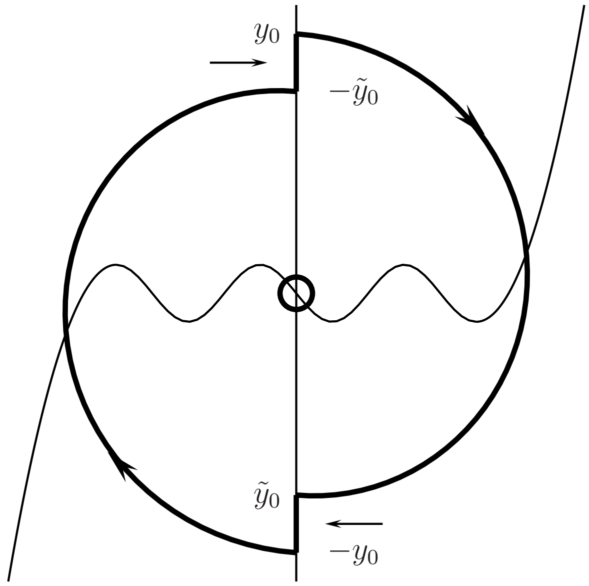
$$\mathcal{S}_3 := \big\{(0, y) \in \mathbb{R}^2 \;\big|\; -\tilde{y}_0 < y < y_0\big\},$$

$$\mathcal{S}_4 := \big\{(0, y) \in \mathbb{R}^2 \;\big|\; -y_0 < y < \tilde{y}_0\big\},$$

and let

$$\mathcal{S}_5 := \big\{(x, y) \in \mathbb{R}^2 \;\big|\; x^2 + y^2 = \varepsilon^2\big\},$$

for some small ε. Then it is not hard to see that $\cup_{i=1}^5 \mathcal{S}_i$ is the boundary of a compact, positively invariant region that does not cotain an equilibrium point.



To verify the claim, we will use the function $R(x, y) := (x^2 + y^2)/2$, and show that if y_0 is large enough (and $\tilde{y}_0$ is as defined above) then

$$R(0, y_0) > R(0, \tilde{y}_0).$$