Functions

1. Graph the function \( f(x) = \frac{\frac{1}{2} x^3 - 3}{3 + x} \). Also graph any horizontal asymptotes and vertical asymptotes.
   Answer:
   Graph \( \frac{\frac{1}{2} x^3 - 3}{3 + x} \)

   \[ \left( x, \frac{\frac{1}{2} x^3 - 3}{3 + x} \right) \]

2. Let \( g(t) = \sqrt{(3 - t)} \) and let \( f(t) = t^{12} \). Find \( g \circ f \). Include the domain of this function.
   Answer:
   \[ g \circ f(t) = \sqrt{(3 - t^{12})} \] You need \( |t| \leq \sqrt[6]{3} \).

3. A function \( f(x) = ax + b \) is called an oblique asymptote for a rational function \( g(x) \) if
   \[ \lim_{x \to \pm \infty} (f(x) - g(x)) = 0 \]
   Graph the following rational function and show the horizontal, vertical and oblique asymptotes also.
   \( -\frac{3x^2 + 3}{2x - 3} \)
   Answer:
   Graph \( -\frac{3x^2 + 3}{2x - 3} \)

   \[ \left( x, -\frac{3}{2} x - \frac{9}{4} - \frac{15}{4(2x - 3)} \right) \]
4. Simplify \( \sin(\arctan x) \).
   Answer:
   \[
   \frac{x}{\sqrt{1 + x^2}}
   \]

5. Simplify \( \cos(\arctan x) \).
   Answer:
   \[
   \frac{1}{\sqrt{1 + x^2}}
   \]

6. Find the decimal .37
   Answer:
   \[
   \frac{37}{99}
   \]

7. Let 0 < \( r \) < 1. Find a formula for \( \sum_{k=1}^{n} 9(1 + r)^{-k} \). Then take the limit as \( n \to \infty \).
   Answer:
   \[
   \sum_{k=1}^{n} 9(1 + r)^{-k} = \left( \frac{1}{1+r} \right)^{n+1} \frac{1+r^k}{r} \] and the limit as \( n \to \infty \) is \( 9/r \).

8. Find \( \lim_{x \to \infty} \left( \sqrt{(1 + 6x + x^2)} - \sqrt{(1 + x)^2} \right) \)
   Answer:
   \[
   \lim_{x \to \infty} \left( \sqrt{(1 + 6x + x^2)} - \sqrt{(1 + x)^2} \right) = 2
   \]

9. Let \( f(x) = 2x^2 + 3x \). Show \( f \) is continuous at every value of \( x \).
   Answer:
   Let \( \varepsilon > 0 \) be given. Then
   \[
   \left| 2x^2 + 3x - (2y^2 + 3y) \right| \leq \left| 2|x + y||x - y| + 3|x - y| \right|
   \leq (6)(|x| + |y|)|x - y| + (6)|x - y| \text{ Then for } |x - y| < 1, \text{ it follows } |y| < |x| + 1. \text{ Therefore, for } |y - x| < 1,
   \[
   \left| 2x^2 + 3x - (2y^2 + 3y) \right| \leq 2(6)(2|x| + 1)|x - y|. \text{ Let } \delta < \min \left( 1, \frac{\varepsilon}{2(6)(2|x| + 1)} \right). \text{ Then if } |y - x| < \delta, \text{ it follows } \left| 2x^2 + 3x - (2y^2 + 3y) \right| < 2(6)(2|x| + 1) \frac{\varepsilon}{2(6)(2|x| + 1)} = \varepsilon \text{ and so } f \text{ is continuous at } x.
   \]

10. Let \( f(x) = 3x^3 + 2 \sin x^2 - 11x \). Explain why there exists a point \( a \) such that \( f(a) = 0 \).
    Answer:
    The function is negative for large negative values of \( x \) and is positive for large positive values. Since the function is continuous, it follows from the intermediate value theorem that there exists a
solution to $f(x) = 0$.

11. Give an example of a function which is continuous at only one point.
Answer:
One such example is

$$f(x) = \begin{cases} 
  x & \text{if } x \text{ is rational} \\
  0 & \text{if } x \text{ is not rational}
\end{cases}$$

12. Find $\lim_{x \to \infty} \frac{3x^2 + 3x + \cos x}{5x^2 + 3x + 4}$.
Answer:
$\frac{3}{5}$

13. Let $f$ be any function defined on the integers. Show that $f$ is continuous.
Answer:
Let $\varepsilon > 0$ be given. Then let $\delta = 1/4$. If $|x - y| < 1/4$ and $x, y$ are integers, then they are the same integer and so $|f(x) - f(y)| = 0 < \varepsilon$.

14. Show that if $\lim_{n \to \infty} a_n = a$, then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} a_k = a.$$  

Answer:

$$\left| \frac{1}{n} \sum_{k=1}^{N} |a_k| + \frac{1}{n} \sum_{k=N+1}^{n} a_k - a \right| = \left| \frac{1}{n} \sum_{k=1}^{N} |a_k| + \frac{n-(N+1)}{n} \frac{1}{n-(N+1)} \sum_{k=N+1}^{n} a_k - a \right|$$

Let $N$ be so large that if $k \geq N$, then $|a_k - a| < \varepsilon/2$. Then for such an $N$, the above is no larger than

$$\frac{1}{n} \sum_{k=1}^{N} |a_k| + \left( \frac{n-(N+1)}{n} \frac{1}{n-(N+1)} \sum_{k=N+1}^{n} a_k - \frac{n-(N+1)}{n} \sum_{k=N+1}^{n} a_k \right)$$

$$+ \frac{1}{n-(N+1)} \sum_{k=N+1}^{n} |a_k - a|$$

$$\left( \frac{n-(N+1)}{n} \frac{1}{n-(N+1)} \sum_{k=N+1}^{n} |a| + \varepsilon/2 \right)$$

$$+ \frac{1}{n-(N+1)} \sum_{k=N+1}^{n} |a_k - a|$$

$$\left( \frac{n-(N+1)}{n} \frac{1}{n-(N+1)} \sum_{k=N+1}^{n} |a| + \varepsilon/2 \right)$$

$$+ \frac{1}{n-(N+1)} \sum_{k=N+1}^{n} \varepsilon/2$$

The first two terms converge to 0 as $n \to \infty$ and the last term equals $\varepsilon/2$. 
15. Suppose for large \( n \), \( a_{n+1} = 9a_n - 8a_{n-1} \). Assuming the limit exists, find all possible values of this limit.

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n}.
\]

Answer: 8, 1

16. Show using the binomial theorem that \( \left( 1 + \frac{4}{n} \right)^n \) is increasing.

Answer:

\[
\left( 1 + \frac{4}{n} \right)^n = \sum_{k=0}^{n} \binom{n}{k} \left( \frac{4}{n} \right)^k = \sum_{k=0}^{n} \frac{4^k}{k!} \frac{n}{n} \frac{(n-1)}{n} \frac{(n-2)}{n} \ldots \frac{(n-k+1)}{n} \\
\leq \sum_{k=0}^{n+1} \frac{4^k}{k!} \frac{n}{n+1} \frac{n}{n+1} \frac{(n-1)}{n+1} \frac{(n-2)}{n+1} \ldots \frac{(n-k+1)}{n+1} = \left( 1 + \frac{4}{1+n} \right)^{n+1}.
\]

17. Show \( \lim_{n \to \infty} \frac{7^n}{n!} = 0 \).

Answer:

For all \( n \) large enough

\[
\left( b^{n+1}/(n+1)! \right)/\left( b^n/(n)! \right) < \frac{1}{2}
\]

and so there exists a constant such that

\[
\frac{b^{n+r}}{(n+r)!} \leq C \frac{1}{2^r}
\]

Hence this converges to 0.

18. Find \( \lim_{n \to \infty} 10^{\frac{1}{n}} \).

Answer:

\( \lim_{n \to \infty} 10^{\frac{1}{n}} = 1 \)

19. Show \( (1 + \frac{1}{n})^n \) is bounded above by 3 and the sequence is increasing.

Answer:

It is true for \( n = 1, 2, 3 \). Let \( n > 3 \). \( (1 + \frac{1}{n})^n = 1 + 1 + \frac{n(n-1)}{2n^2} + \sum_{k=3}^{n} \binom{n}{k} \frac{1}{n^k} \)

\[
\leq 2.5 + \sum_{k=3}^{n} \frac{1}{k} \leq 2.5 + \sum_{k=3}^{n} \frac{1}{k(k-1)}.
\]

Now \( \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k} \) and so this sum reduces to \( \frac{1}{2} \).

Hence 3 is an upper bound as claimed. Why is the sequence increasing?

\[
\left( 1 + \frac{1}{n} \right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} \\
\leq \sum_{k=0}^{n+1} \frac{n+1}{n+1} \frac{n}{n+1} \frac{n+2-k}{n+1} \frac{1}{k!} \\
= \left( 1 + \frac{1}{n+1} \right)^{n+1}
\]
20. Show that for large $n$ and $b$ a positive integer, \( \left(1 + \frac{b}{n}\right)^n \leq 3^b \).

Answer:

You first show that \( n \to \left(1 + \frac{b}{n}\right)^n \) is increasing. Then note that

\[
(1 + \frac{1}{n})^{(b)(n)} = \left(1 + \frac{b}{(b)(n)}\right)^{(b)(n)} \leq 3^b.
\]

Thus \( \left(1 + \frac{b}{n}\right)^n \leq \left(1 + \frac{b}{b\cdot n}\right)^{b\cdot n} \leq 3^b \).