Objectives for math 214  
Text: Calculus, by L. Garner

Chapter 11 Vectors and Vector Functions

11.1 Vectors in Two and Three Dimensions

1. Know what rectangular coordinates mean. They can be set up for the plane (two dimensions) or for three-dimensional space. Later we’ll use two additional coordinate systems for three dimensions: cylindrical and spherical.

2. Know the distance formula between two points.

3. Think of a vector as an arrow. In this course, a vector can represent a point, a displacement, a force, a velocity, or an acceleration. You want to be familiar with how to do algebra with vectors (add, subtract, etc.), and also be familiar with how each of the algebraic operations relates geometrically to points, displacements, forces, etc.

4. Understand that “Arrows that are the same length and point in the same direction represent the same vector. That is to say, the location of an arrow does not affect the vector it represents; conversely, a vector may be represented by any arrow that has the right length and direction, placed in whatever location may be convenient.” (p. 761).

5. Be familiar with the following representations:
   If a vector is being used to represent a given point, then think of the tail of the arrow being at the origin, and the head of the arrow being at the given point.
   If a vector is representing a displacement, then the tail is placed at the original location and the head is then at the new location. Know the notation \( \vec{v}(PQ) \), which denotes the vector with tail at the point \( P \) and head at the point \( Q \).
   If a vector represents a force, then the arrow’s length represents how strong the force is, and the arrow’s direction represents the direction in which the force is acting.
   If a vector represents a velocity, then the arrow’s length and direction give the speed and direction of the velocity.
   A vector representing acceleration shows how fast and in what way velocity is changing.

6. Know the meanings of “coordinates” of a vector, and “position vector” (or “position arrow”).

7. Know how to find the magnitude of a vector, and what is meant by a “unit vector.”

8. Know both the algebra and the geometry of scalar multiplication of a vector, norm of a vector, addition of two vectors, and subtraction of two vectors.

9. Memorize the triangle inequality (11.7).

10. Understand two notations for a vector: \( \langle 2, 3, 4 \rangle = 2\vec{i} + 3\vec{j} + 4\vec{k} \), for example. This is because we define \( \vec{i} = \langle 1, 0, 0 \rangle \), so \( 2\vec{i} = \langle 2, 0, 0 \rangle \). Similarly \( 3\vec{j} = \langle 0, 3, 0 \rangle \) and \( 4\vec{k} = \langle 0, 0, 4 \rangle \),
so that \(2\vec{i} + 3\vec{j} + 4\vec{k} = \langle 2, 3, 4 \rangle\). Don’t make the mistake of mixing the two notations and writing something like \(\langle 2\vec{i}, 3\vec{j}, 4\vec{k} \rangle\); this isn’t a vector.

11.2 Vector Products

Dot product

1. Know the simple formula for dot product: For example,

\[ \langle 1, 2, 3 \rangle \cdot \langle -1, 3, 5 \rangle = (1)(-1) + (2)(3) + (3)(5) = -1 + 6 + 15 = 20. \]

Be sure you know that the dot product takes two vectors and produces a real number. Don’t make the mistake of thinking that \(\langle 1, 2, 3 \rangle \cdot \langle -1, 3, 5 \rangle \) is \(\langle -1, 6, 15 \rangle\)!

2. Know the properties

11.21 (dot product is commutative),
11.22 (you can factor scalars out of a dot product), and
11.23 (dot product is distributive).

3. Know the geometry you can obtain from the dot product: lengths, angles, projections, and components; see objectives (4) through (7) below.

4. Know the formula

11.24 \(\vec{u} \cdot \vec{v} = \|\vec{v}\|^2\).

Thus, the square root of a vector dot itself is the length of the vector. This is significant when you want to work with vectors without breaking them into their components; be able to switch between the two expressions in the twinkling of an eye.

5. Know the formula \(\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta\), where \(\theta\) is the angle between the vectors \(\vec{u}\) and \(\vec{v}\). Thus, to find the angle, you can use the formula

\[ \theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}\right). \]

In particular, note that \(\vec{u}\) is perpendicular to \(\vec{v}\) if and only if \(\vec{u} \cdot \vec{v} = 0\).

6. Given two vectors \(\vec{u}\) and \(\vec{v}\), know how to find the component of \(\vec{v}\) in the direction of \(\vec{u}\). This is the scalar

\[ \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \vec{v}}. \]

(Cross-check: Notice that if we double \(\vec{u}\), i.e., multiply it by 2, this doesn’t change what we get from the formula for the component of \(\vec{v}\) in the direction of \(\vec{u}\), which is what we would want. On the other hand, if we double \(\vec{v}\), the formula gives an answer twice as big, which is also what we want.)

7. Closely related to the component of \(\vec{v}\) in the direction of \(\vec{u}\) is the projection of \(\vec{v}\) onto \(\vec{u}\). Know the formula:

\[ \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} \vec{u}. \]
(Cross-check: Note what happens in the formula if we double \( \vec{u} \) or double \( \vec{v} \). Is it what you would expect?)

**Cross product**

8. Know that the cross product is defined only for three-dimensional vectors. It takes two vectors and gives back a vector. (For two-dimensional vectors, if you tack on a zero for the third component of each, you can then take the cross product.)

9. Know the **geometry** of the cross product: The vector \( \vec{u} \times \vec{v} \) is a vector whose **direction** is perpendicular to both \( \vec{u} \) and \( \vec{v} \), and whose **length** equals the area of the parallelogram spanned by \( \vec{u} \) and \( \vec{v} \). Both of these facts will be very useful to us. Know also how to use the right-hand rule to find which of the two possible directions \( \vec{u} \times \vec{v} \) could point in (since there are two directions perpendicular to the plane spanned by \( \vec{u} \) and \( \vec{v} \)).

10. Know the **algebra** of the cross product:
\[
\vec{u} \times \vec{v} = -\vec{v} \times \vec{u},
\]
\[
\vec{u} \times \vec{u} = (0, 0, 0),
\]
\[
\vec{u} \times (\vec{v} + \vec{c}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{c},
\]
\[
(\vec{u} + \vec{v}) \times \vec{c} = \vec{u} \times \vec{c} + \vec{v} \times \vec{c}.
\]
The cross product is distributive, *anticommutative*, and not associative at all.

Know also the relation
\[
\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin \theta,
\]
where \( \theta \) is the angle between \( \vec{u} \) and \( \vec{v} \).

11. Know the **formula** for cross product, given in three alternative forms in (11.28), (11.29), and (11.30). In class we’ll use (11.30).

12. Know the formula and geometry of the triple product \( \vec{u} \times \vec{v} \cdot \vec{w} \).

**10.3 Planes in Space**

1. Know that the (scalar) equation for the plane in space which contains the point \( \langle x_0, y_0, z_0 \rangle \) and which is perpendicular to the vector \( \langle A, B, C \rangle \) is
\[
A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.
\]

2. Know how to rewrite the above in vector notation:
\[
\mathbf{N} \cdot (\vec{r} - \vec{r}_0) = 0
\]
where \( \mathbf{N} = \langle A, B, C \rangle \), \( \vec{r} = \langle x, y, z \rangle \), and \( \vec{r}_0 = \langle x_0, y_0, z_0 \rangle \).

3. Given a scalar equation for a plane (objective 1 above), be able to find the two unit normal vectors to the plane. (Here normal means perpendicular.)

4. Become familiar with the often-useful idea of thinking about planes by thinking about their normals. For example, the angle between two intersecting planes is the same as the
angle between their normals. In particular, two planes are perpendicular if and only if their normals are perpendicular to each other. Another characterization of perpendicular planes is that one plane contains the normal vector of the other.

5. Given three points, be able to find the equation for the plane containing them.

**11.4 Surfaces**

1. Recognize that when an equation for a surface in space is missing a variable, then the surface is a “cylinder” parallel to the axis of that variable.

2. Be able to list the nine types of quadric surfaces:

<table>
<thead>
<tr>
<th>Elliptic Cylinder</th>
<th>Hyperbolic Cylinder</th>
<th>Parabolic Cylinder</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elliptic Paraboloid</td>
<td>Hyperbolic Paraboloid</td>
<td></td>
</tr>
<tr>
<td>Elliptic Cone</td>
<td>Hyperboloid of one sheet</td>
<td></td>
</tr>
<tr>
<td>Ellipsoid</td>
<td>Hyperboloid of two sheets</td>
<td></td>
</tr>
</tbody>
</table>

3. Given an equation for a quadric surface, be able to identify which quadric surface it is.

**11.5 Lines in Space**

1. Here we point out the important difference between an equation and a parametrization. (At first it may be confusing to distinguish because both use the “=” sign.) For example, a line in the plane may be expressed by an equation such as

\[ 2x + 3y = 12, \]

or by a parametrization such as

\[ x = 3t, y = 4 - 2t. \]

The parametrization may be also be expressed as

\[ f(t) = (x(t), y(t)) = (2t, 4 - t). \]

The equation and the parametrization give the same line, but in different ways.

In the first expression, think of \( x \) and \( y \) as the “inputs;” among all possible inputs \( \langle x, y \rangle \) you are picking out only the ones that satisfy the equation.

In the second expression, think of \( t \) as the input and \( x \) and \( y \) as the outputs; now instead of picking out certain inputs, we use all values of \( t \), and calculate \( x \) and \( y \) from \( t \).

2. A line in the plane is usually described by an equation. For a line in space we can either use two equations, such as the symmetric form (11.54), or we use a parametrization.

3. Know the notation for the “scalar parametric equations” for a line in space; these are simply the individual coordinates of the vector parametrization.

4. Know how to find where two lines in space intersect (if they do at all).
5. Given two intersecting lines written in parametric form, know how to find the angle between them.

### 11.6 Curves in Space

1. Understand how a vector function \( \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \) describes a curve in space. Namely, for each \( t \), the output of \( \mathbf{r}(t) \) is a vector, which we place as a position vector (i.e., starting at the origin), and let the curve consist of the endpoints of these position vectors.

2. Know how to take the derivative \( \mathbf{r}'(t) \), namely, it is just \( \langle x'(t), y'(t), z'(t) \rangle \). Similarly, \( \mathbf{r}''(t) = \langle x''(t), y''(t), z''(t) \rangle \), etc.

3. Given a curve and a point on the curve, be able to find the tangent line at that point.

4. Given a curve \( \mathbf{r}(t) \), be able to calculate the arc length. In many cases the integral can’t be done because no antiderivative can be found, in which case just be able to set up the integral for arc length.

### 11.7 Curvilinear Motion

1. Understand that \( \mathbf{r}(t) \) is position, \( \mathbf{r}'(t) \) is velocity, and \( \mathbf{r}''(t) \) is acceleration of an object in motion. Also, \( \| \mathbf{r}'(t) \| \) is the speed.

2. Given a curve \( \mathbf{r}(t) \) representing uniform circular motion, be able to find angular velocity. (The coefficient of \( t \) in \( \langle a \cos(\omega t), a \sin(\omega t) \rangle \).

3. Know the formulas acceleration = \( r \omega^2 = v^2/r \), where the motion is uniform circular, and \( r \) is the radius, \( \omega \) the angular velocity, and \( v \) the speed.

4. Be able to work a problem like exercise #7 (p. 814) on a test.

### 11.8 Curvature

1. Know the intuitive meaning of curvature, which can be expressed in various ways: How fast the curve is turning, or how fast the tangent vector is turning, or how much you’d have to turn your wheels to drive along the curve, or (proportional to) how much you’d get pushed against the side door of the car if you drove at unit speed along the curve. (The last two assume that the curve is in the plane)

   Note: The curvature \( \text{vector } \kappa(t) \) is a vector whose length is the curvature \( \kappa(t) \). It is perpendicular to the curve, in the direction that the tangent vector is moving toward.

2. Know that a circle has constant curvature. If the radius of the circle is \( a \), then the curvature is \( \kappa = 1/a \). (The Greek letter \( \kappa \) is pronounced “kappa.”)

3. Be able to calculate curvature of a curve using each of the formulas (11.80,11.81), (11.94), and (11.96). (The last one works only in the plane.)

4. Understand the following terms and notations: unit tangent vector \( T(t) \), circle of curvature, radius of curvature \( \rho(t) \), center of curvature \( \mathbf{r}(t) + \rho(t)^2 \kappa(t) \), and osculating plane. Be able to find all of these at a given point of a given curve.
5. Be able to find the unit normal to a curve.

11.9 Review Homework: Work all that you need to work to master the objectives. Don’t turn it in.