Harmonic mappings onto parallel slit domains

by Michael Dorff (Provo, UT), Maria Nowak (Lublin) and Magdalena Wołoszkiewicz (Lublin)

Abstract. We consider typically real harmonic univalent functions in the unit disk \( D \) whose range is the complex plane slit along infinite intervals on each of the lines \( x \pm ib \), \( b > 0 \). They are obtained via the shear construction of conformal mappings of \( D \) onto the plane without two or four half-lines symmetric with respect to the real axis.

1. Introduction. Let \( S_H \) be the class of functions \( f \) that are univalent sense-preserving harmonic mappings of the unit disk \( \mathbb{D} = \{ z : |z| < 1 \} \) and satisfy \( f(0) = 0 \) and \( f_z(0) > 0 \). Next let \( S^0_H \) be the subclass of \( S_H \) consisting of \( f \) with \( f_z(0) = 0 \). Since harmonic mappings in \( S^0_H \) are not determined by their image domains, many authors have studied subclasses of \( S^0_H \) consisting of functions mapping \( \mathbb{D} \) onto a specific simply connected domain \( \Omega \). In particular, in [6] Hengartner and Schober considered the case of \( \Omega \) being the horizontal strip \( \{ w : |\text{Im} w| < \pi/4 \} \). Later Dorff [2] considered the case of \( \Omega \) being an asymmetric vertical strip, and Livingston [7] considered the case of \( \Omega \) being the plane \( \mathbb{C} \) slit along the interval \( (a, \infty) \), \( a < 0 \). Also Livingston [8], and Szapiel and Grigoryan [5] studied the case when \( \Omega \) is \( \mathbb{C} \setminus (\infty, a] \cup [b, \infty) \).

Here we consider the case when a simply connected domain \( \Omega \) is the plane slit along infinite intervals on each of the lines \( x \pm ib \) with some \( b > 0 \). Let \( S^R_H(\mathbb{D}, \Omega) \subset S^0_H \) be the class of harmonic typically real functions \( f \) mapping the disk \( \mathbb{D} \) onto \( \Omega \). Since the domain \( \Omega \) is convex in the horizontal direction, as in the cases mentioned above, the shear construction introduced by Clunie and Sheil-Small can be applied. In our case the so-called conformal preshear \( Q \) is typically real and maps the disk onto the plane without two or four half-lines symmetric with respect to the real axis. In the next section we study the properties of the function \( Q \) and, in particular, we find the preimages of horizontal lines \( \text{Im} Q = \alpha \). We also define a family \( \mathcal{F} \) of harmonic

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mappings such that \( S_R^H(\mathbb{D}, \Omega) \subset \mathcal{F} \). We discuss properties of functions from the family \( \mathcal{F} \) and present several examples of harmonic functions from \( \mathcal{F} \).

2. Conformal preshear. We start with the following

**Lemma 2.1.** For \( A, B > 0 \) and \( c \in [-2, 2] \), the function \( Q(z) \) defined by
\[
Q(z) = A \log \frac{1+z}{1-z} + B \frac{z}{1+cz+z^2}
\]
is a univalent map of \( \mathbb{D} \) onto a domain convex in the direction of the real axis.

**Proof.** We will show that \( iQ(z) \) maps \( \mathbb{D} \) onto a domain convex in the direction of the imaginary axis. By the result of Royster and Ziegler \cite{9}, it suffices to show that there are numbers \( \mu \in [0, 2\pi), \gamma \in [0, \pi] \), such that
\[
\text{Re}\left\{e^{i\mu}(1 - 2 \cos \gamma e^{-i\mu}z + e^{-2i\mu}z^2)Q'(z)\right\} \geq 0, \quad z \in \mathbb{D}.
\]
Choosing \( \mu = 0 \) and \( \gamma \in [0, \pi] \) so that
\[
\cos \gamma = -c/2 \in [-1, 1] \implies \text{the left-hand side of the last inequality is equal to}
\]
\[
\text{Re}\left\{(1 + cz + z^2) \left(2A \frac{1}{1-z^2} + B \frac{1-z^2}{(1+cz+z^2)^2}\right)\right\}
\]
\[
= \left(\frac{2A}{|1-z^2|^2} + \frac{B}{|1+cz+z^2|^2}\right)(1-|z|^2)(1+|z|^2 + c\text{Re}(z)).
\]
So the result follows from the fact that \( c \in [-2, 2] \).

We remark that in the case when \( A = \frac{1}{2} \sin^2 \alpha, B = \cos^2 \alpha, \alpha \in (0, \pi/2) \), and \( c = -2 \), Lemma 2.1 was proved in \cite{4} where the authors also studied classes of harmonic mappings obtained by shearing these functions.

A calculation shows that in the case of \( c = 2 \) the image of the unit disk under \( Q \) is
\[
\mathbb{C} \setminus \left\{x \pm \frac{A\pi}{2} i : x \in \left[-\frac{A}{2} \log \frac{2A+B}{4}, \infty\right)\right\},
\]
while for \( c = -2 \) the image is
\[
\mathbb{C} \setminus \left\{x \pm \frac{A\pi}{2} i : x \in \left(-\infty, \frac{A}{2} \log \frac{2A-B}{4} - \frac{2A+B}{4}\right)\right\}.
\]
In the case when \( c \in (-2, 2) \) the function \( Q \) maps the unit disk onto the complex plane minus four horizontal half-lines. In particular, if \( c = 0 \), then the resulting image is the \( \mathbb{C} \) plane without the four symmetric half-lines
\[
\left\{x \pm \frac{A\pi}{2} i : x \in \left(-\infty, -\frac{A}{2} \log \left(\frac{\sqrt{2A+B} + \sqrt{B}}{\sqrt{2A+B} - \sqrt{B}}\right) - \frac{\sqrt{B(2A+B)}}{2}\right)\right\}.
\]
and
\[ \left\{ x \pm \frac{A\pi}{2} i : x \in \left[ \frac{A}{2} \log \left( \frac{\sqrt{2A + B + \sqrt{B}}}{\sqrt{2A + B - \sqrt{B}}} \right) + \frac{\sqrt{B(2A + B)}}{2}, \infty \right) \right\}. \]

Assume now that \( Q \) is given by (2.1) with \( c = -2 \cos \gamma, \gamma \in (0, \pi) \). Then, setting \( \eta = e^{i\gamma} \), we have
\[
Q(z) = A \log \frac{1 + z}{1 - z} + B \frac{z}{(1 - \eta z)(1 - \overline{\eta}z)}.
\]

Our aim is now to study the preimages of the horizontal lines \( \text{Im} \ Q = \alpha > 0 \). Using the transformation \( \zeta = \zeta(z) = \frac{1 + z}{1 - z} \) we can write
\[
Q(z) = A \log \zeta + B \frac{\zeta^2 - 1}{4 \sin^2 \frac{\gamma}{2} (\zeta + i \cot \frac{\gamma}{2})(\zeta - i \cot \frac{\gamma}{2})}.
\]

We put \( \zeta = re^{i\theta} \) and consider the level curve
\[
\text{Im} \ Q = A\theta + \frac{B}{4 \sin^4 \frac{\gamma}{2}} \left( \frac{\sin 2\theta}{(r - \frac{\cot^2(\gamma/2)}{r})^2 + 4 \cot^2 \frac{\gamma}{2} \cos^2 \theta} \right) = \alpha,
\]
where
\[ 0 < \theta < \min\{\alpha/A, \pi/2\}. \]

So, the equations of these level curves in polar coordinates can be written in the form
\[
\left( r - \frac{\cot^2 \frac{\gamma}{2}}{r} \right)^2 = \frac{B \sin 2\theta}{4(\alpha - A\theta) \sin^4 \frac{\gamma}{2}} - 4 \cot^2 \frac{\gamma}{2} \cos^2 \theta,
\]
or
\[
\left( r + \frac{\cot^2 \frac{\gamma}{2}}{r} \right)^2 = \frac{B \sin 2\theta}{4(\alpha - A\theta) \sin^4 \frac{\gamma}{2}} + 4 \cot^2 \frac{\gamma}{2} \sin^2 \theta.
\]

Consequently,
\[
(2.3) \quad r - \frac{\cot^2 \frac{\gamma}{2}}{r} = \pm 2 \cot \frac{\gamma}{2} \cos \theta \sqrt{\frac{B \tan \theta}{2(\alpha - A\theta) \sin^2 \gamma} - 1} = \pm \cot \frac{\gamma}{2} \sqrt{\frac{B \sin 2\theta}{(\alpha - A\theta) \sin^2 \gamma} - 4 \cos^2 \theta},
\]
and

\[
(2.4) \quad r + \frac{\cot^2 \frac{\gamma}{2}}{r} = 2 \cot \frac{\gamma}{2} \sin \theta \sqrt{\frac{B \cot \theta}{2(\alpha - A\theta) \sin^2 \gamma}} + 1
\]

\[
= \cot \frac{\gamma}{2} \sqrt{\frac{B \sin 2\theta}{(\alpha - A\theta) \sin^2 \gamma}} + 4 \sin^2 \theta.
\]

We assume first that \(\alpha > \pi A/2\) and show that preimage of \(\Im Q = \alpha\) in the \(z\)-plane is a Jordan curve passing through the point \(\eta\) and except for this point lying in the upper half of \(\mathbb{D}\). It follows from (2.3) that \(\theta \in (\theta_0, \pi/2)\), where \(\theta_0\) satisfies the equation

\[
\frac{B \tan \theta}{2(\alpha - A\theta) \sin^2 \gamma} = 1.
\]

It follows from (2.3) and (2.4) that

\[
(2.5) \quad r = \frac{1}{2} \cot \frac{\gamma}{2} \left( \sqrt{\frac{B \sin 2\theta}{(\alpha - A\theta) \sin^2 \gamma}} + 4 \sin^2 \theta \right)
\]

\[
\pm \sqrt{\frac{B \sin 2\theta}{(\alpha - A\theta) \sin^2 \gamma} - 4 \cos^2 \theta},
\]

where \(\theta \in (\theta_0, \pi/2)\). On the other hand,

\[
(2.6) \quad \Re Q = A \log r
\]

\[
+ \frac{B}{4 \sin^2 \frac{\gamma}{2}} \left( r - \frac{\cot (\gamma/2)}{r} \right) \left( r + \frac{\cot (\gamma/2)}{r} \right) + \cos 2\theta \left( \cot^2 \frac{\gamma}{2} - 1 \right).
\]

It follows from the above that the first term in (2.6) is bounded and a calculation gives that the second term is equal to

\[
(2.7) \quad \frac{1}{2 \sin^2 \gamma} \left( B \cos \gamma 
\right.
\]

\[
\pm \sqrt{B + 2(\alpha - A\theta) \sin^2 \gamma \tan \theta \sqrt{B - 2(\alpha - A\theta) \sin^2 \gamma \cot \theta}}.
\]

This shows that \(\Re Q\) tends to \(\pm \infty\) if \(\theta\) tends to \(\pi/2\), which means that the preimage of the level curve \(\Im Q = \alpha\) in the \(\zeta\)-plane is a Jordan curve passing through the point \(i \cot (\gamma/2)\) lying in the first quadrant except for this point and our claim is proved.

Assume now that \(0 < \alpha < A\pi/2\). Then the preimage of the level curve \(\Im Q = \alpha\) in the \(\zeta\)-plane in polar coordinates is also given by (2.5), where \(\theta \in (\theta_0, \alpha/A)\). This implies that if \(\theta\) tends to \(\alpha/A\), then \(r\) tends to either 0 or \(\infty\). Moreover, by (2.7) the second term in the sum on the right-hand side of equation (2.6) is bounded for \(\theta \in (\theta_0, \alpha/A)\). This means that the preimage
of the level curve $\text{Im } Q = \alpha$ in the $\zeta$-plane is a regular line going from zero to infinity which corresponds to a curve connecting 1 and $-1$ in the upper half of $\mathbb{D}$ in the $z$-plane.

Finally we note that the preimage of an interval lying on the line $\text{Im } w = A\pi/2$ is a curve joining two boundary points of $\mathbb{D}$ where the derivative of $Q$ vanishes.

We have already mentioned that in the case when $c = 2, -2$, the function $Q$ maps the unit disk onto the plane slit along two parallel horizontal half-lines. In the manner used above but with less tedious calculations one can show that in these cases preimages of the horizontal lines $\text{Im } Q = \alpha$ are curves connecting 1 and $-1$ for $0 < \alpha < A\pi/2$ and Jordan curves passing through $-1$ (resp. 1) for $\alpha > A\pi/2$.

3. The class $S^R_H(\mathbb{D}, \Omega)$. Let $\Omega$ and $S^R_H(\mathbb{D}, \Omega)$ be as in the Introduction and assume that $f \in S^R_H(\mathbb{D}, \Omega)$. Next, let $F$ and $G$ be functions analytic in $\mathbb{D}$ satisfying

$$F(0) = G(0) = 0, \quad \text{Re } f(z) = \text{Re } F(z), \quad \text{Im } f(z) = \text{Im } iG(z).$$

If

$$h = (F + iG)/2 \quad \text{and} \quad g = (F - iG)/2$$

then

$$f = h + \bar{g} \quad \text{and} \quad |g'(z)| < |h'(z)|.$$ 

Moreover, the function $h - g = iG$ is univalent, convex in the horizontal direction, and $G(\mathbb{D})$ is $\mathbb{C}$ slit along one or two infinite rays on the vertical lines $x = \pm b$. We also note that $f$ is typically real if and only if $iG = h - g$ is typically real. So the image of $\mathbb{D}$ under $iG$ is symmetric with respect to the real axis.

It follows from the above that

$$iG(z) = Q(z) = A \log \frac{1 + z}{1 - z} + B \frac{z}{1 + cz + z^2},$$

where $A, B > 0$, $c \in [-2, 2]$. We also note that $A = 2b/\pi$.

Consequently,

$$F(z) = h(z) + g(z) = \int_0^z \frac{h'(\zeta) + g'(\zeta)}{h'(\zeta) - g'(\zeta)} (h'(\zeta) - g'(\zeta)) d\zeta = \int_0^z iG'(\zeta) P(\zeta) d\zeta,$$

where $P$ is in the class $\mathcal{P}$ of functions analytic in $\mathbb{D}$ with $P(0) = 1$ and $\text{Re } P(z) > 0$ for $z \in \mathbb{D}$. 


Thus
\[
f(z) = \text{Re} \left\{ \int_0^z \left( \frac{A(1-z^2) + B(1+cz+\zeta^2)}{1-\zeta^2} \right) P(\zeta) \, d\zeta \right\}
+ i \text{Im} \left\{ A \log \frac{1+z}{1-z} + B \frac{z}{1+cz+z^2} \right\}.
\]

Using the function
\[
Q_{A,B,c}(z) = A \log \frac{1+z}{1-z} + B \frac{z}{1+cz+z^2}
\]
the last formula can be written in the form
\[
(3.1) \quad f(z) = \text{Re} \int_0^z Q_{A,B,c}(\zeta) P(\zeta) \, d\zeta + i \text{Im} Q_{A,B,c}(z).
\]

Now we define the family
\[
\mathcal{F} = \left\{ f: f(z) = \text{Re} \int_0^z Q_{A,B,c}(\zeta) P(\zeta) \, d\zeta + i \text{Im} Q_{A,B,c}(z), \quad A, B > 0, \ c \in [-2, 2], \ P \in \mathcal{P} \right\}.
\]

So, we have

**Theorem 3.1.** \( S^R_{\mathcal{F}}(\mathbb{D}, \Omega) \subset \mathcal{F} \).

The next theorem gives one of the properties of the family \( \mathcal{F} \) that can be proved using the method applied by Hengartner and Schober \[6\] and Grigorian and Szapiel \[5\] and others. We include its proof for the reader’s convenience.

**Theorem 3.2.** For each \( f \in \mathcal{F} \), every horizontal line has a non-empty connected intersection with the image \( f(\mathbb{D}) \).

**Proof.** Let \( f \in \mathcal{F} \), \( f = h + \bar{g} = \text{Re}(h + g) + i \text{Im}(h - g) \). Let \( \Omega = Q(\mathbb{D}) \). We consider the images of horizontal lines contained in \( \Omega \) under the function \( f \circ Q^{-1} \). We observe that in the case when \( \alpha \neq \pm b \) the entire line \( \{w = t + i\alpha : t \in \mathbb{R}\} \) is contained in \( \Omega \) while \( \{w = t \pm ib : t \in \mathbb{R}\} \cap Q(\mathbb{D}) \) are finite or infinite intervals. Note first that
\[
\text{Im}[f(Q^{-1}(t+i\alpha))] = \text{Im}[Q(Q^{-1}(t+i\alpha))] = \alpha,
\]
so the function \( f \circ Q^{-1} \) maps horizontal lines into themselves. Moreover,
\[
\frac{\partial}{\partial t} [f(Q^{-1}(t+i\alpha))] = \frac{\partial}{\partial t} [\text{Re}(f(Q^{-1}(t+i\alpha)))]
= \text{Re} (Q'(Q^{-1}(t+i\alpha))P(Q^{-1}(t+i\alpha))Q^{-1}(t+i\alpha)')
= \text{Re} (P(Q^{-1}(t+i\alpha))) > 0.
\]
Thus the functions $t \mapsto \Re(f \circ Q^{-1}(t + i\alpha))$ are strictly increasing for each $\alpha \in \mathbb{R}$. Therefore every horizontal line has a non-empty intersection with $f(\mathbb{D})$.

In the next theorem we give some sufficient conditions for the containment of the entire horizontal lines $\Im z = \alpha$ ($\alpha \neq \pm b$) in $f(\mathbb{D})$.

**Theorem 3.3.** Assume that $Q$ is given by (2.2) with $\eta = e^{i\gamma}$ and $f$ is defined by (3.1). Let $\gamma \in [0, \pi]$. If the function $P$ in (3.1) is analytic at $\eta$ and $\Re P(\eta) > 0$, then the half-plane $\{w : \Im w > b\}$ is contained in $f(\mathbb{D})$. If the function $P$ is analytic at $\bar{\eta}$ and $\Re P(\bar{\eta}) > 0$, then the half-plane $\{w : \Im w < -b\}$ is contained in $f(\mathbb{D})$. Finally, if the function $P$ is analytic at 1 and $-1$, $\Re P(1) > 0$ and $\Re P(-1) > 0$, then the horizontal strip $\{w : |\Im w| < b\}$ is contained in $f(\mathbb{D})$.

**Proof.** Assume $P$ is analytic at $\eta$ and $\Re P(\eta) > 0$. Consider the function

$$
F(z) = \int_0^z Q'(\zeta)P(\zeta) \, d\zeta,
$$

where $Q$ is given by (2.2). Then in a neighborhood of $\eta$, when $\eta \neq \pm 1$,

$$
F'(z) = P(\eta)Q'(z) + \left(P'(\eta)(z - \eta) + \frac{P''(\eta)}{2}(z - \eta)^2 + \cdots\right)
$$

$$
\times \left(-\frac{B\eta}{(\eta - \bar{\eta})(z - \eta)^2} + \frac{a_{-1}}{z - \eta} + a_0 + \cdots\right),
$$

and when $\eta^2 = 1$,

$$
F'(z) = P(\eta)Q'(z) + \left(P'(\eta)(z - \eta) + \frac{P''(\eta)}{2}(z - \eta)^2 + \cdots\right)
$$

$$
\times \left(-\frac{2B\eta}{(z - \eta)^3} - \frac{B}{(z - \eta)^2} + \frac{a_{-1}}{z - \eta} + a_0 + \cdots\right).
$$

Thus the function $w_{\eta}$ defined by

$$
w_{\eta}(z) = \begin{cases} 
F(z) - P(\eta)Q(z) + \frac{B\eta P'(\eta)}{\eta - \bar{\eta}} \log(1 - \bar{\eta}z) & \text{if } \eta^2 \neq 1, \\
F(z) - P(\eta)Q(z) & \text{if } \eta^2 = 1,
\end{cases}
$$

is analytic at $\eta$. Consequently, in the case $\eta^2 \neq 1$,

$$
F(z) = F(z) - w_{\eta}(z) + w_{\eta}(z)
$$

$$
= Q(z) \left(P(\eta) - \frac{B\eta P'(\eta)(1 - \eta z)(1 - \bar{\eta}z) \log(1 - \bar{\eta}z)}{(\eta - \bar{\eta})(1 - \eta z)(1 - \bar{\eta}z) \log(1 - \bar{\eta}z)}\right) + w_{\eta}(z),
$$
and in the case $\eta^2 = 1$,

$$F(z) = Q(z)\left(P(\eta) + \frac{B\left((P'(\eta) + \eta P''(\eta)) \log \frac{1}{1-\eta z} - \frac{2P'(\eta)}{1-\eta z}(1 - \eta z)^2\right)}{A(1-\eta z)^2 \log \frac{1+z}{1-z} + Bz}\right) + w_\eta(z).$$

Therefore,

$$F(z) = Q(z)(P(\eta) + o(1)) + w_\eta(z) \quad \text{as} \quad z \to \eta.$$ 

It follows from the work in Section 2 that the preimages $\Gamma_\alpha$ of the lines

$$\text{Im } f(z) = \text{Im } Q(z) = \alpha > b \quad \text{or} \quad \text{Im } f(z) = \text{Im } Q(z) = \alpha < -b$$

are curves in $\mathbb{D}$ that approach $\eta$ or $\bar{\eta}$, respectively. Since

$$\text{Re } f(z) = \text{Re } F(z),$$

we see that $\text{Re } f(z)$ converges to $\pm \infty$ as $z$ approaches $\eta$ or $\bar{\eta}$ along $\Gamma_\alpha$.

Assume now that $\eta = e^{i\gamma}$ with $\gamma \in (0, \pi)$. If the function $P$ is analytic at 1 and $-1$, $\text{Re } P(1) > 0$, and $\text{Re } P(-1) > 0$, then $w_1(z) = F(z) - P(1)Q(z)$ is analytic at 1 and $w_{-1}(z) = F(z) - P(-1)Q(z)$ is analytic at $-1$. This means that $\text{Re } f(z) = \text{Re } F(z)$ behaves as $\text{Re } Q(z)$ near 1 and $-1$. Moreover, we know from Section 2 that preimages of the lines

$$\text{Im } f(z) = \text{Im } Q(z) = \alpha, \quad \text{where } |\alpha| < b,$$

are curves in $\mathbb{D}$ connecting 1 and $-1$. So, our claim follows. The same conclusion can be drawn for the cases when $\eta = 1$ and $\eta = -1$.]

**Corollary 3.4.** If $f \in \mathcal{F}$ has dilatation $\omega(z) = g'(z)/h'(z)$ such that $|\omega(z)| \leq C < 1$ for $z \in \mathbb{D}$, then the complement of $f(\mathbb{D})$ consists of infinite intervals lying on two parallel lines $z = \pm ib$.

For fixed $A, B > 0$, $c \in [-2, 2]$ let $\mathcal{F}(A, B, c)$ denote the subset of $\mathcal{F}$ with $Q = Q_{A,B,c}$. As we noted before, the class $\mathcal{F}(A, B, c)$ contains the harmonic univalent maps of the disk $\mathbb{D}$ onto the plane slit along the horizontal lines $z = \pm ib$, where $b = \pi A/2$. Now for fixed $b > 0$ (or equivalently $A > 0$) let

$$\mathcal{F}(b) = \bigcup_{B > 0, -2 \leq c \leq 2} \mathcal{F}(A, B, c)$$

and let $S_H^R(b)$ denote the class of typically real univalent harmonic mappings of the disk $\mathbb{D}$ onto the plane slit along the horizontal lines $z = \pm ib$. We have the following.

**Corollary 3.5.** For $b > 0$,

$$\overline{S_H^R(b)} = \mathcal{F}(b).$$
Proof. Let \( f \in \mathcal{F}(b) \) be given by (3.1) with some \( P \in \mathcal{P} \). For an integer \( n > 2 \) define \( P_n(z) = P((1 - 1/n)z) \) and set

\[
f_n(z) = \Re \int_0^z Q'(\zeta) P_n(\zeta) \, d\zeta + i \Im Q(z).
\]

By Theorem 3.3, \( f_n \in S_H^R(b) \) and the sequence \( \{f_n\} \) converges locally uniformly on \( \mathbb{D} \) to \( f \). ■

The next theorem describes situations when functions \( f \) from the family \( \mathcal{F} \) have the property that the intersections of horizontal lines with \( f(\mathbb{D}) \) are finite intervals.

**Theorem 3.6.** Assume that \( Q \) is given by (2.2) with \( \eta = e^{i\gamma}, \gamma \in (0, \pi) \), and \( f \) is defined by (3.1). If the function \( P \) in (3.1) is analytic at \( \eta \) (\( \bar{\eta} \)) and \( P(\eta) = 0 \) (\( P(\bar{\eta}) = 0 \)), then the intersection of every horizontal line \( \Im w = \alpha \), \( \alpha > b \) (\( \alpha < -b \)), with \( f(\mathbb{D}) \) is a finite interval. Moreover, if the function \( P \) is analytic at 1 and \( -1 \), and \( P(1) = P(-1) = 0 \), then the intersection of a horizontal line \( \Im w = \alpha \) (\( |\alpha| < b \)) with \( f(\mathbb{D}) \) is a finite interval.

**Proof.** Assume that \( P \) is analytic at \( \eta \), \( P(\eta) = 0 \) and \( F \) is given by (3.2). Then in a neighborhood of \( \eta \),

\[
F'(z) = -\frac{B\eta P'(\eta)}{(\eta - \bar{\eta})(z - \eta)} + w_\eta(z),
\]

where \( w_\eta \) is analytic at \( \eta \). Consequently,

\[
F(z) = \frac{B\eta P'(\eta)}{\eta - \bar{\eta}} \log \frac{1}{1 - \bar{\eta}z} + W_\eta(z),
\]

with \( W_\eta \) analytic at \( \eta \). It has been noted in [3], pp. 66–67 that \( \eta P'(\eta) < 0 \). Hence in a neighborhood of \( \eta \),

\[
\Re f(z) = \Re F(z) = \Im \left( \frac{B\eta P'(\eta)}{2 \sin \gamma} \log \frac{1}{1 - \bar{\eta}z} \right) + \Re W_\eta(z).
\]

Now our claim follows from the properties of the set \( \{ z \in \mathbb{D} : \Im f(z) = \alpha \} \) for \( \alpha > b \). The other statement can be proved by observing that if \( P \) is analytic at 1 and \( -1 \), and \( P(1) = P(-1) = 0 \), then \( F \) is analytic at 1 and \( -1 \). ■

We note that the assertion of Theorem 3.6 does not hold in the case \( \eta = \pm 1 \). In particular, if \( \eta = 1 \), \( P \) is analytic at 1 and \( P(1) = 0 \), then the intersection of every horizontal line \( \Im w = \alpha \) (\( \alpha > b \)) with \( f(\mathbb{D}) \) is either this line or a half-line \( \{ w : w = x + i\alpha, \ x > x_\alpha \} \) with some real \( x_\alpha \). Indeed, if

\[
Q(z) = A \log \frac{1 + z}{1 - z} + B \frac{z}{(1 - z)^2}
\]
and $F$ is defined by (3.2), then
\[
F(z) = \frac{2BP'(1)}{(z - 1)} + B(P'(1) + P''(1)) \log \frac{1}{z - 1} + w(z),
\]
where $w$ is analytic at 1. Hence
\[
\Re F(z) = 2BP'(1) \Re \frac{1}{z - 1} + B(P'(1) + \Re P''(1)) \log \frac{1}{|z - 1|} + O(1)
\]
as $D \ni z \to 1$. Using the transformation $\zeta = \frac{1 + z}{1 - z}$ we can write
\[
\Re F(\zeta) = -BP'(1) \Re \zeta + B(P'(1) + \Re P''(1)) \log |\zeta + 1| + O(1) \text{ as } \zeta \to \infty.
\]
A calculation shows that the preimage of the level curve $\Im f = \Im Q = \alpha > b$ in the $\zeta$-plane can be written in the form
\[
(3.3) \quad r = 2\sqrt{\frac{\alpha - A\theta}{B}} \sin 2\theta,
\]
where $\zeta = re^{i\theta}$, $\theta \in (0, \pi/2)$. It has been proved in \cite{5} that $P'(1) + \Re P''(1) \leq 0$. We now show that if we assume additionally that $P'(1) + \Re P''(1) = 0$, then $f(\mathbb{D})$ contains the half-lines described above. Indeed, on the curve given by (3.3) we have
\[
\Re F(\zeta) = -BP'(1) \cdot 2\sqrt{\frac{\alpha - A\theta}{B}} \cos \theta + O(1)
\]
and our claim follows from the fact that
\[
\lim_{\theta \to 0^+} 2\sqrt{\frac{\alpha - A\theta}{B}} \cos \theta = +\infty \quad \text{and} \quad \lim_{\theta \to \pi/2^-} 2\sqrt{\frac{\alpha - A\theta}{B}} \cos \theta = 0.
\]
Similar analysis can be used to show that if $P'(1) + \Re P''(1) < 0$, then $f(\mathbb{D})$ contains the whole horizontal lines $\Im w = \alpha > b$.

4. Examples. In this section we give examples of harmonic functions from the family $\mathcal{F}$. Our first example is a harmonic map of the unit disk onto the complex plane slit along four horizontal half-lines that are symmetric with respect to the real axis.

Example 4.1. Let $Q_1 = Q_{1/4,1/2,0}$ and take $P(z) = \frac{1 + z^4}{1 - z^4}$. Then we obtain
\[
f_1(z) = \Re F_1(z) + i \Im Q_1(z)
\]
\[
= \Re \left( -\frac{5i}{16} \log \left( \frac{1 + iz}{1 - iz} \right) + \frac{1}{4} \frac{z}{1 - z^2} - \frac{1}{8} \frac{z}{1 + z^2} + \frac{1}{4} \frac{z}{(1 + z^2)^2} \right)
\]
\[
+ i \Im \left( \frac{1}{4} \log \left( \frac{1 + z}{1 - z} \right) + \frac{1}{2} \frac{z}{1 + z^2} \right).
\]
We will show that the function $f_1$ maps the unit disk onto the plane minus four parallel slits given by \{x \pm i\pi/8 : |x| \geq 5\pi/32\}. We will use a similar
argument to that applied by Clunie and Sheil-Small [1] for the so-called harmonic Koebe function. Using the transformation ζ = ζ(z) = \frac{1+z}{1-z} = \xi + i\eta, \xi > 0, we get

\begin{align*}
f_1(z) &= \text{Re} \left( -\frac{5i}{16} \log \left( \frac{\zeta - i}{1 - i\zeta} \right) + \frac{1}{16} \left( \zeta - \frac{1}{\zeta} \right) + \frac{1}{8} \frac{(\zeta^2 - 1)\zeta}{(\zeta^2 + 1)^2} \right) \\
&\quad + i \text{Im} \left( \frac{1}{4} \log \zeta + \frac{1}{4} \frac{\zeta^2 - 1}{\zeta^2 + 1} \right).
\end{align*}

We observe that the transformation \( z \mapsto \zeta(z) \) maps the part of the disk in the first quadrant onto the exterior of the unit disk contained in the first quadrant, and we note that the interval \([0, i]\) is mapped onto the quarter of the unit circle. If we put \( \zeta = re^{i\theta}, r \geq 1, \theta \in [0, \pi/2) \), then we have

\begin{align*}
\text{Re} f_1(z) &= \frac{1}{4} \left( \frac{5}{4} \arctan \frac{r - 1/r}{2\cos \theta} + \frac{1}{4} \left( r - \frac{1}{r} \right) \cos \theta \\
&\quad + \frac{1}{2} \left( r - \frac{1}{r} \right) \cos \theta \frac{(r - 1/r)^2 + 4(\sin^2 \theta + 1)}{(r - 1/r)^2 + 4 \cos^2 \theta} \right), \\
\text{Im} f_1(z) &= \frac{1}{4} \left( \theta + \frac{2 \sin 2\theta}{(r - 1/r)^2 + 4 \cos^2 \theta} \right).
\end{align*}

Now we consider the level curves

\begin{equation}
\theta + \frac{2 \sin 2\theta}{(r - 1/r)^2 + 4 \cos^2 \theta} = c, \quad c > 0.
\end{equation}

Since \( r > 1 \) and \( \theta \in (0, \pi/2) \), we get

\begin{equation}
r - \frac{1}{r} = 2 \cos \theta \sqrt{\tan \theta \frac{\tan \theta}{c - \theta} - 1}.
\end{equation}

Let \( \theta_c \in (0, \pi/2) \) be the number satisfying the equation \( \tan \theta_c = c - \theta_c \). If \( 0 < c < \pi/2 \), we assume that \( \theta_c < \theta < c \), while if \( c \geq \pi/2 \), we assume that \( \theta_c < \theta < \pi/2 \). Fix \( c > 0 \). Then the image of the level curve given in (4.1) under \( f_1 \) is

\begin{align*}
f_1(z) &= \frac{1}{8} \left( \frac{5}{2} \arctan \left( \frac{\tan \theta}{c - \theta} - 1 \right) \right)^{1/2} + \cos^2 \theta \left( \frac{\tan \theta}{c - \theta} - 1 \right)^{1/2} \\
&\quad + \frac{1}{2} \left( c - \theta \right)^2 \left( \frac{\tan \theta}{c - \theta} - 1 \right)^{3/2} \\
&\quad + \frac{1}{2} \left( c - \theta \right)^2 \left( 1 + \frac{1}{\sin^2 \theta} \right) \left( \frac{\tan \theta}{c - \theta} - 1 \right)^{1/2} \right) + \frac{i}{4} \frac{c}{4} \\
= u(c, \theta) + \frac{i}{4} \frac{c}{4}.
\end{align*}
If \(0 < c < \pi/2\), then \(\theta \in (\theta_c, c)\) and we find that
\[
\lim_{\theta \to \theta_c^+} u(c, \theta) = 0 \quad \text{and} \quad \lim_{\theta \to c^-} u(c, \theta) = \infty.
\]
Similarly, if \(c > \pi/2\), then \(\theta \in (\theta_c, \pi/2)\) and we have
\[
\lim_{\theta \to \theta_c^+} u(c, \theta) = 0 \quad \text{and} \quad \lim_{\theta \to \pi/2^-} u(c, \theta) = \infty.
\]
Finally, if \(c = \pi/2\), then \(\theta \in (\theta_c, \pi/2)\) and we have
\[
\lim_{\theta \to \theta_c^+} u(c, \theta) = 0 \quad \text{and} \quad \lim_{\theta \to \pi/2^-} u(c, \theta) = \frac{5\pi}{32}.
\]
This means that the image under \(f_1\) of the part of the disk in the first quadrant is the first quadrant minus the half-line \(\{x + i\pi/8 : x \geq 5\pi/32\}\).

Our claim follows from the symmetry.

In the next example we present a map onto the plane slit along two horizontal half-lines symmetric with respect to the real axis.

**Example 4.2.** Let \(f_2\) be the harmonic shear of \(Q_2 = Q_{1/8,6/8,-2}\) with \(P(z) = (1 + z^2)/(1 - z^2)\). One can show that
\[
f_2(z) = \Re F_2(z) + i \Im Q_2(z)
= \Re \left( \frac{1}{2} \frac{z(2 - z + z^3)}{(1 - z)^3(1 + z)} \right) + i \Im \left( \frac{1}{8} \log \frac{1 + z}{1 - z} + \frac{6}{8} \frac{z}{(1 - z)^2} \right).
\]
It was shown in [3] that \(f_2\) maps the disk onto the plane minus two half-lines given by \(x \pm i\pi/16, x \leq -1/4\).

The following two examples illustrate Theorem 3.6.

**Example 4.3.** Taking \(Q_3 = Q_{1/4,1/2,0}\) and \(P(z) = (1 - z^2)/(1 + z^2)\) we obtain
\[
f_3(z) = \Re \left( -\frac{3i}{8} \log \frac{1 + iz}{1 - iz} - \frac{1}{4} \frac{z}{1 + z^2} + \frac{1}{2} \frac{z}{(1 + z^2)^2} \right)
+ i \Im \left( \frac{1}{4} \log \frac{1 + z}{1 - z} + \frac{1}{2} \frac{z}{1 + z^2} \right).
\]

**Example 4.4.** Let \(f_4\) be the shear of \(Q_4 = Q_{1/4,1/2,0}\) with \(P(z) = (1 - z^4)/(1 + z^4)\). Then
\[
f_4(z) = \Re \left( -\frac{i}{2} \log \frac{1 + iz}{1 - iz} \right) + i \Im \left( \frac{1}{4} \log \frac{1 + z}{1 - z} + \frac{1}{2} \frac{z}{1 + z^2} \right).
\]
Images of concentric circles inside \(\mathbb{D}\) under \(f_3\) and \(f_4\) are shown in the figures below.

Our final example is a harmonic map onto the right-half plane. This map is connected with the note after Theorem 3.6.
Harmonic mappings onto parallel slit domains

Example 4.5. Let $Q_5 = Q_{1/4,1/2,-2}$ and take $P(z) = (1 - z^2)/(1 + z^2)$. Then

$$f_5(z) = \text{Re} \left( \frac{z}{1 - z} \right) + i \text{Im} \left( \frac{1}{4} \log \frac{1 + z}{1 - z} + \frac{1}{2} \frac{z}{(1 - z)^2} \right)$$

is the harmonic map of the disk onto the half-plane $\text{Re} w > -1/2$.

References


Michael Dorff
Department of Mathematics
Brigham Young University
Provo, UT 84602, U.S.A.
E-mail: mdorff@math.byu.edu

Maria Nowak, Magdalena Wołoszkiewicz
Department of Mathematics
Maria Curie-Skłodowska University
20-031 Lublin, Poland
E-mail: mt.nowak@poczta.umcs.lublin.pl
woloszkiewicz@umcs.pl

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