Convolutions of harmonic convex mappings

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Communicated by W. Koepf

(Received 29 December 2009; final version received 17 April 2010)

The first author proved that the harmonic convolution of a normalized right half-plane mapping with either another normalized right half-plane mapping or a normalized vertical strip mapping is convex in the direction of the real axis, provided that it is locally univalent. In this article, we prove that in general the assumption of local univalency cannot be omitted. However, we are able to show that in some cases these harmonic convolutions are locally univalent. Using this we obtain interesting examples of univalent harmonic maps one of which is a map onto the plane with two parallel slits.

Keywords: harmonic mappings; convolutions; univalence

AMS Subject Classification: 30C45

1. Introduction

Let $D$ be the unit disc. We consider the family of complex-valued harmonic functions $f = u + iv$ defined in $D$, where $u$ and $v$ are real harmonic in $D$. Such functions can be expressed as $f = h + g$, where

$$h(z) = \sum_{n=1}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n$$

are analytic in $D$. The harmonic function $f = h + g$ is locally one-to-one and sense-preserving in $D$ if and only if

$$j g^{\prime}(z) g(z) \leq j h^{\prime}(z) h(z) \quad \forall z \in D$$

Let $S^o$ be the class of complex-valued, harmonic, sense-preserving, univalent functions $f$ in $D$, normalized so that $f(0) = 0$, $f'(0) = 1$ and $f(\neq 0) \neq 0$. Let $K^o_h$ and $C^o_h$ be the subclasses of $S^o$ mapping $D$ onto convex and close-to-convex domains, respectively. We will deal with $C^o_h$ mappings that are convex in one direction.
For analytic functions $f \cdot h \in \mathbb{D}^1$ and $g \cdot h \in \mathbb{D}^1$, their convolution (or Hadamard product) is defined as $f \cdot F = \sum a_n z^n = \sum b_n z^n$. In the harmonic case, with

$$f \cdot F = \sum a_n z^n \cdot \sum b_n z^n$$

and

$$F \cdot H = \sum A_n z^n \cdot \sum B_n z^n,$$

define the harmonic convolution as

$$f \cdot F = \sum a_n A_n z^n \cdot \sum b_n B_n z^n.$$
Next, we deal mainly with the convolution of the canonical harmonic right half-plane mapping [1] given by

\[
f_0 \circ z^b \quad \text{with harmonic mappings f that are either right half-planes or strip mappings. We show that if the dilatation of f is } e^{i\theta} z^n (n \neq 1, 2), \text{ then } f_0 \circ f \text{ is locally univalent. However, we give examples when local univalency fails for } n = 3. \text{ Also, we provide some results about univalency in the case the dilatation of f is } e^{i\theta} z^n. \text{ Finally, we give examples of univalent harmonic maps obtained by way of convolutions.}
\]

2. The convolution of slanted half-plane mappings

We first prove a generalization of Theorem A for the slanted half-plane, \( H_y \), \( 0 < y < 2\pi n \), described in the introduction. Let \( S^0 \circ H_y \cap S^0 \) denote the class of harmonic functions \( f \) that map \( D \) onto \( H_y \). In the case when \( y = 0 \) we get the normalized class of harmonic functions that map \( D \) onto the right half-plane \( \{ w : \text{Re}(w) \geq 1/2 \} \).

**Lemma 1** If \( f = h + g \in S^0 \circ H_y \), then

\[
h \circ z^b \circ e^{-2iy} g \circ z^{-b} \quad \text{is convex in the direction } n/2 - y \text{.}
\]

**Proof** If \( f = h + g \in S^0 \circ H_y \), then \( \text{Re} (e^{iy} (z) h(z) + e^{-iy} g(z)) = 4 - 1/2 \). In other words, \( \text{Re} (e^{iy} (h(z) e^{-2iy} g(z))) = 4 - 1/2 \). Since \( f \) is a convex harmonic function, it follows from a convexity theorem by Clunie and Sheil-Small [1] that \( f(z) e^{-2iy} g(z) \) is convex in the direction \( n/2 - y \), and so is univalent. It is also clear that \( z \circ h(z) \circ e^{-2iy} g(z) \) maps \( D \) onto \( H_y \) which implies the result.

**Theorem 2** If \( f_k \in S^0(H_y) \), \( k \neq 1, 2 \), and \( f_1 \circ f_2\) is locally univalent in \( D \), then \( f_1 \circ f_2\) is convex in the direction \( -(y_1 \circ y_2) \).

**Proof** Let

\[
F_1 = h_1 \circ e^{-2y_1} g_1 \circ \quad F_2 = h_2 \circ e^{-2y_2} g_2 \circ \quad \text{and}
\]

\[
\frac{1}{2} F_1 \circ F_2 \quad \text{where } \frac{1}{2} F_1 \circ F_2 \quad \text{and } \frac{1}{2} F_1 \circ F_2 \quad \text{are done.}
\]

The shearing theorem of [1] establishes that it is sufficient to show that the function \( \frac{1}{2} F_1 \circ F_2 \) is convex in the direction \( -(y_1 \circ y_2) \), or equivalently, that \( F \circ e^{iy} \circ (F_1 \circ F_2) \) is convex in the direction of the real axis. A result by Royster and Ziegler [8] shows that \( F \) is convex in the real direction, if \( \text{Re}(z^F \circ (F_1 \circ F_2)) \circ 4 \circ 0 \circ z^D \), where \( \delta \circ \frac{d}{d_1 - ze^{a^b}} \), with some \( a \in R \). Thus, if we show this last condition, we are done.
By Lemma 1,
\[
\mathcal{H}^\theta_0(z) \quad \frac{z}{1 - ze^{-iy}} \quad [z\partial h_2 - e^{-2iy} g_2 h_0^\theta z] .
\]
Furthermore,
\[
\mathcal{H} h_2 - e^{-2iy} g_2 h_0^\theta z \quad \frac{1}{4} z \left( \frac{h_2^\theta z - e^{-2iy} g_2^\theta z}{h_2^\theta z - e^{-2iy} g_2^\theta z} \right) \frac{1}{4} \left( \frac{1 - e^{-iy} z}{1 - e^{-iy} z} \right) \left( \frac{h_2^\theta z - e^{-2iy} g_2^\theta z}{h_2^\theta z - e^{-2iy} g_2^\theta z} \right) \frac{z p_2^\theta z}{\partial 1 - e^{iy} z} ;
\]
Since \( e^{-iy} z \in \mathbb{D} \) on \( D \) and \( e^{-iy} z \neq 0 \), if we let \( p_2^\theta z \quad \frac{1}{4} \left( \frac{1 - e^{-iy} z}{1 - e^{-iy} z} \right) \) then we have that \( \text{Re}\{p_2(z)\} \neq 0 \) for all \( z \in D \). Consequently,
\[
\mathcal{H}^\theta_0(z) \quad \frac{z}{1 - ze^{-iy}} \quad \frac{z p_2^\theta z}{\partial 1 - e^{iy} z} ;
\]
Analogously,
\[
\mathcal{H}^\theta_0(z) \quad \frac{z}{1 - ze^{-iy}} \quad \frac{z p_2^\theta z}{\partial 1 - e^{iy} z} ;
\]
where \( \text{Re}\{p_1(z)\} \neq 0 \) for all \( z \in D \). Thus
\[
0 \quad \frac{1}{e^{iy} z \partial F^0_0 z} \quad \frac{z p_2^\theta z}{\partial 1 - e^{iy} z} ;
\]
This completes the proof.

3. The convolution of \( f_0 \) with right half-plane mappings

In Theorems A, B, and 2, we require that the resulting convolution function is locally univalent and sense-preserving. That is,
\[
\mathcal{H}^\theta_{z} \quad \frac{1}{h^\theta z} \mathcal{K} \quad \frac{1}{h^\theta z} \mathcal{K} 1 \quad \text{with} \ h^\theta z 6 \quad \frac{1}{4} \quad 8 \ z \in D .
\]
When is this not a necessary assumption? In the rest of this article we establish cases of these theorems for which this assumption can be omitted.

The following result about the number of zeros of polynomials in the disc is helpful in proving the next several theorems.
Cohn’s Rule ([9] or see [10, p. 375]) Given a polynomial
\[ f(z) = a_0 + a_1 z + \cdots + a_n z^n \]
of degree \( n \), let
\[ f' \frac{\partial z}{\partial b} \]
of degree \( n - 1 \) with \( p_1 \) \( \frac{3}{4} p - 1 \) and \( s_1 \) \( \frac{3}{4} s \) the number of zeros of \( f' \) inside the unit circle and on it, respectively. If \( j a_0 j \leq j a_n j \), then
\[ f_1(z) = \frac{a_0 f(z) - a_n f(z)}{z} \]
is of degree \( n - 1 \) and \( p_1 = p - 1 \) and \( s_1 = s \) the number of zeros of \( f_1 \) inside the unit circle and on it, respectively.

The main result of this section is the following.

**Theorem 3** Let \( f \) be an analytic function in \( D(1) \) such that \( f(z) = h(z) + g(z) \) with \( h(z) = z \) and \( g(z) = L \). If \( n \) is an integer, then \( f(z) \) is convex in the direction of the real axis.

Proof Let the dilatation of \( f \) be given by \( \delta = \frac{g_0 + h_0}{h_0 + g_0} \). By Theorem A and by Lewy’s theorem, we just need to show that \( f(z) = 1 \) on \( \{ 1 \} \). By Equation (3)

\[ h_0 z + g_0 z = h_0 + g_0 \]

Also, since \( g(z) \) is \( \frac{3}{4} \) \( (z)h'(z) \), we know \( g_0(z) \) is \( \frac{3}{4} \) \( (z)h_0'(z) \). Hence

\[ \delta = \frac{g_0(z)}{h_0(z)} \]

Using \( h(z) = z \) and \( g_0(z) \) is \( \frac{3}{4} \) \( (z)h_0(z) \), we can solve for \( h'(z) \) and \( h_0'(z) \) in terms of \( z \) and \( (z) \):

\[ h_0'(z) = \frac{1}{z} \]

Substituting these formulae into the equation for \( \delta \), we derive:

\[ \delta = \frac{z h_0'(z)'}{h_0(z)} \]
Now, consider the case in which \( \lambda(z) = \frac{1}{4} e^{i\theta} z \). Then Equation (6) yields

\[
\mathbf{d}z \mu = \frac{1}{4} - \frac{z e^{i\theta} z}{z} = \frac{1}{4} - \frac{e^{i\theta} z}{z}.
\]

Note that \( q(z) = \frac{1}{4} z^2 - \frac{1}{4} z \). In such a situation, if \( z_0 \) is a zero of \( p \), then \( \frac{1}{z_0} \) is a zero of \( q \). Hence,

\[
\mathbf{d}z \mu = \frac{1}{4} - \frac{z e^{i\theta} z}{z} = \frac{1}{4} - \frac{e^{i\theta} z}{z}.
\]

It suffices to show that \( jA, jB \cdot \cdot \cdot 1 \). We will use Cohn's rule to do this, although the results can be obtained in other ways. Note that

\[
p(z) = \frac{1}{4} - \frac{z e^{i\theta} z}{z} = \frac{1}{4} - \frac{e^{i\theta} z}{z}.
\]

Hence, \( p \) has one zero at \( z_0 \), and so by Cohn's rule \( p \) has two zeros, namely \( A \) and \( B \), in \( D \).

Next, consider the case in which \( \lambda(z) = \frac{1}{4} e^{i\theta} z^2 \). In this case,

\[
j \lambda dz b \mu = \frac{1}{4} - \frac{z e^{i\theta} z}{z} = \frac{1}{4} - \frac{e^{i\theta} z}{z}.
\]

Remark 1 If we assume the hypotheses of the previous theorem with the exception that \( n \geq 3 \), then for some value of \( z \in D \), \( j \lambda dz b \mu = 1 \). To see this, suppose this is not true. Then letting \( \lambda(z) = \frac{1}{4} z^n \), Equation (6) yields

\[
\mathbf{d}z \mu = \frac{1}{4} - \frac{z e^{i\theta} z}{z} = \frac{1}{4} - \frac{e^{i\theta} z}{z}.
\]

The function \( R \) preserves symmetry about the unit circle, because \( jR(e^{i\theta}) \cdot \frac{1}{4} \) and \( 1 = R(1/z) \cdot \frac{1}{4} R(z) \). So, \( R \) maps the closed unit disc onto itself. Hence, \( R \) is a finite Blaschke product of order \( n \cdot 1 \). However, \( \frac{1}{2} \) is the product of the moduli of the zeros of \( R \) in the unit disc. This is a contradiction since \( n \geq 3 \).

Theorem 4 Let \( f = h + g \in K^\alpha \) with \( h \lambda dz b \mu \geq 0 \) and \( \lambda dz b \mu \cdot \frac{1}{4} - \lambda (1-z) \), \( h = h_{1-az} \lambda dz b \mu \cdot \frac{1}{4} - \lambda (1-z) \), \( h_{1-az} \lambda dz b \mu \cdot \frac{1}{4} - \lambda (1-z) \).

Then \( f \) is convex in the direction of the real axis.

Proof Using Equation (6) with \( \lambda \cdot \frac{1}{4} - \lambda (1-z) \), \( h_{1-az} \lambda dz b \mu \cdot \frac{1}{4} - \lambda (1-z) \), \( h_{1-az} \lambda dz b \mu \cdot \frac{1}{4} - \lambda (1-z) \), we have

\[
\mathbf{d}z b \mu = \frac{1}{4} - \frac{z e^{i\theta} z}{z} = \frac{1}{4} - \frac{e^{i\theta} z}{z}.
\]

Again using Cohn's rule,

\[
f_{1} \lambda dz b \mu = \frac{1}{4} - \frac{z e^{i\theta} z}{z} = \frac{1}{4} - \frac{e^{i\theta} z}{z}.
\]

So \( f \) has one zero at \( z_0 \), which is in the unit circle since \( -1 \leq \lambda \leq 1 \). Thus, \( jA, jB \leq 1 \).
Next, we provide some examples.

Example 1  Let \( f_1 = h_1 + g_1 \), where 
\[
\begin{align*}
    h_1 &= \frac{1}{4} \log \frac{1 - z}{1 - z} + \frac{1}{4} \log \frac{1 - z}{1 - z}, \\
    g_1 &= -\frac{1}{4} \log \frac{1 - z}{1 - z} + \frac{1}{4} \log \frac{1 - z}{1 - z}.
\end{align*}
\]

Consider \( F_1 = \frac{1}{4} f_0 \) \( f_1 \frac{1}{4} H_1 \) \( \frac{1}{4} G_1 \). Using Equation (4) we have
\[
\begin{align*}
    H_1 &= h_1 + h_1', \\
    G_1 &= g_1 + g_1'.
\end{align*}
\]

We show that \( F_1 \) maps the unit disc onto the domain whose boundary consists of the four half-lines given by \( \Re z \leq 0 \), \( \Re z \geq \frac{1}{2} \) and \( \Im z \leq \frac{1}{2} \) (Figure 1). In doing so, we use a similar argument to that used by Clunie and Sheil-Small in Example 5.4 of [1]. We have
\[
F_1 z = z - \frac{1}{2} z^2 - \frac{1}{2} z^3 + \frac{1}{4} \log \frac{1 - z}{1 - z} + \frac{1}{4} \log \frac{1 - z}{1 - z}.
\]
Applying the transformation \( \phi = \frac{1}{4} \left( \frac{1}{1-z} \right) \), we get

\[
F_1(\phi) = \frac{1}{8} \left( 3\phi - 2 - \frac{1}{\phi} \right) \text{Re} + \frac{i}{4} \text{Im} - \ln \left| \phi \right| - \frac{1}{2} \phi^2 - 1.
\]

\[
\frac{1}{8} \left( 3\phi - 2 - \frac{1}{\phi} \right) \frac{1}{\phi} \arctan \left( \frac{1}{\phi} \right) + i.
\]

Observe first that the positive real axis \( \{ \phi = \frac{1}{4} \phi i \ i \ : \ i \in [0, 4] \} \) is mapped monotonically onto the whole real axis. Next we find the images of the level curves

\[ \arctan \frac{1}{\phi} \phi i 4 c, \quad \phi, i \in [0, 4]: \]

The polar coordinates equations of these level curves are

\[ B \phi r^2 \sin B \cos \frac{1}{4} c, \quad 0 \leq B \leq \frac{n}{2}; \]

Hence

\[ R_{\phi} = \frac{1}{4} \phi \frac{1}{1-z} \frac{1}{\phi} B \cot B, \quad 0 \leq B \leq \frac{n}{2}; \]

\[ B = \frac{1}{4} \phi \frac{1}{1-z} \frac{1}{\phi} \frac{1}{\phi} \frac{1}{\phi} c. \]

Fix \( c = 0 \). Then the image of the curve given in (7) under \( F_1 \) is

\[ F_1(\phi) = \frac{1}{8} \left( 3\phi - 2 - \frac{1}{\phi} \right) \text{Re} + \frac{i}{4} \text{Im} - \ln \left( \phi \right) - \frac{1}{2} \phi^2 - 1. \]

If \( 0 \leq c \leq \frac{n}{2} \), then \( B = (0, c) \), and one easily finds that \( \lim_{B \to B} u(c, B) = \frac{1}{4} 1 \) and \( \lim_{B \to \infty} u(c, B) = \frac{1}{4} 1 \). The intermediate value property implies that in this case the image of the level curve under \( F_1 \) is the entire horizontal line \( \{ x + ic : -1 \leq x \leq 1 \} \). If \( c = \frac{n}{2} \), then \( \lim_{B \to \infty} u(c, B) = \frac{1}{4} 1 \) and \( \lim_{B \to \infty} u(c, B) = \frac{1}{4} 1 \). So in this case the images of the level curves are horizontal half-lines \( \{ x + ic : -1 \leq x \leq 1 \} \). This means that images of the level curves under \( F_1 \) fill the domain whose boundary consists of the real axis and two half-lines \( \{ x + ic : -1 \leq x \leq 1 \} \). Finally, our claim follows from the fact that the range of \( F_1 \) is symmetric with respect to the real axis.

The images of concentric circles inside \( D \) under the harmonic maps \( f_0 \) and under \( f_1 \) are shown in Figure 2. The images of these concentric circles under the convolution map \( f_0 f_1 \) are shown in Figure 1.

Example 2 Let \( f_2 \) be the harmonic mapping in the disc \( D \) such that

\[ h_2(\phi) = \frac{1}{4} \frac{1}{1-z} \frac{1}{1-z} \frac{1}{1-z} \frac{1}{1-z} \frac{1}{1-z}. \]

One can find that

\[ h_2(\phi) = \frac{1}{8} \left( 3\phi - 2 - \frac{1}{\phi} \right) \text{Re} + \frac{i}{4} \text{Im} - \ln \left( \phi \right) - \frac{1}{2} \phi^2 - 1. \]

\[ g_2(\phi) = \frac{1}{8} \left( 3\phi - 2 - \frac{1}{\phi} \right) \frac{1}{\phi} \arctan \left( \frac{1}{\phi} \right) + i. \]
and the image of $D$ under $f_2$ is the right half-plane, $R_{\frac{1}{4}} \subset \{ \text{Re} z \} < -\frac{1}{2}$. We note here that $f_2(e^{i\theta} \bar{z}) \approx -\frac{1}{2} + i \frac{n}{16}$, if $0 < \theta < \frac{n}{16}$ and $f_2(e^{i\theta} \bar{z}) \approx -\frac{i}{2} + i \frac{n}{16}$, if $\frac{16}{3} \pi < \theta < \frac{16}{3} \pi \frac{n}{6}$. Next let

$$F_2 \left[ \frac{1}{4} h_0 \right] \left[ \frac{1}{4} g_0 \right] \left[ \frac{1}{4} H_2 \right] \bar{G}_2 :$$

By Equation (4)

$$H_2(\bar{z}) = 2 \ln \frac{1}{1 - z} + 2 \ln \frac{1}{1 + z} + 4 \left[ (1 - z)^2 + (1 - z)^3 (1 + z) \right],$$

and

$$G_2(\bar{z}) = 2 \ln \frac{1}{1 - z} - 2 \ln \frac{1}{1 + z} - 4 \left[ (1 - z)^2 + (1 - z)^3 (1 + z) \right].$$

Analysis similar to that in Example 1 can be used to show that $F_2$ maps the disc onto the plane minus two half-lines given by $x = \pm \frac{16}{3} \pi$, $x \in \mathbb{R}$. We have

$$F_2(\bar{z}) = \text{Re} \left( \frac{1}{2} \ln \frac{1 - z}{1 + z} + \frac{1}{2} \ln \frac{1 - z}{1 + z} \right) + \frac{1}{2} \ln \frac{1 - z}{1 + z} + \frac{1}{2} \ln \frac{1 - z}{1 + z},$$

which under the same transformation as in Example 1 becomes

$$F_2(\bar{z}) = \text{Re} \left( \frac{1}{2} \ln \frac{1 - z}{1 + z} + \frac{1}{2} \ln \frac{1 - z}{1 + z} \right) + \frac{1}{2} \ln \frac{1 - z}{1 + z} + \frac{1}{2} \ln \frac{1 - z}{1 + z},$$

Analogously, we find that the images of the level curves

$$B \left[ \frac{3}{2} r^2 \sin 2 \theta \right] \left[ \frac{1}{4} c \right] \left[ \frac{n}{2} \right] \subset \mathbb{S} \subset \mathbb{S} \subset \mathbb{S}$$
are
\[
F_{2}dzb^\frac{1}{4} \frac{1}{16} \left( \frac{1}{3} \cot B - \tan B \right) \frac{1}{2} \cot B - \tan B \]
\[
\frac{1}{3} \cot B - \tan B \]
\[
= 4 - \sin 2B \frac{1}{2} \cot B - \tan B - 4 \frac{1}{8} \frac{i}{c}
\]
\[
\frac{1}{4} u dc, Bb \frac{i}{8} c:
\]
If \(0 \leq c \leq \frac{n}{2}\) (or \(c \geq 4 \frac{n}{2}\) respectively), then \(\lim_{B \to 0^+} u(c, B) \to 1\) (or \(\lim_{B \to 0^+} u(c, B) \to 1\), respectively) and \(\lim_{B \to 0^+} u(c, B) \to \frac{n}{2}\). So, \(F_{2}\) maps the first quadrant onto the upper half-plane minus the half-line \(fx + i\frac{n}{2}: x \geq \frac{n}{2}\), and the result follows from the symmetry.

4. The convolution of \(f_{0}\) with vertical strip mappings

In this section we replace right half-plane maps with vertical strip maps and prove the corresponding analogues for Theorems 3 and 4.

**Theorem 5** Let \(f \frac{1}{4} h b g 2 K_{H}^{O}\) with \(h dz b \frac{i}{g dz b} \frac{1}{4} \frac{1}{2} \sin a \log \frac{1}{1 + ze^{-ia}}\), where \(\frac{n}{2} \ldots a \leq n\) and \(f(z) \frac{1}{4} e^{ib} z^{2}\). If \(n \frac{1}{4} 1, 2\), then \(f_{0} f 2 S_{i}^{O}\) and is convex in the direction of the real axis.

**Proof** By Theorem B we need to establish that \(f_{0} f \frac{1}{4} H b G \) is locally univalent. Using \(h dz b \frac{i}{g dz b} \frac{1}{4} \frac{1}{2} \sin a \log \frac{1}{1 + ze^{-ia}}\) and \(g'(z) \frac{1}{4} f'(z)h'(z)\), we get

\[
h' \frac{1}{4} \frac{1}{2} \frac{2}{1 + ze^{-ia}} - \frac{1}{1 + ze^{-ia}}
\]
\[
g' \frac{1}{4} \frac{1}{2} \frac{2}{1 + ze^{-ia}} - \frac{1}{1 + ze^{-ia}}
\]
\[
= \left( \frac{1}{1 + ze^{-ia}} - \frac{1}{1 + ze^{-ia}} \right)
\]
\[
= \left( \frac{1}{1 + ze^{-ia}} - \frac{1}{1 + ze^{-ia}} \right)
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\[
= \left( \frac{1}{1 + ze^{-ia}} - \frac{1}{1 + ze^{-ia}} \right)
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\]
\[
= \left( \frac{1}{1 + ze^{-ia}} - \frac{1}{1 + ze^{-ia}} \right)
\]

First, consider the case in which \(f(z) \frac{1}{4} e^{ib} z.\) We have

\[
d \frac{1}{4} \frac{1}{2} \frac{2}{1 + ze^{-ia}} - \frac{1}{1 + ze^{-ia}}
\]
\[
d \frac{1}{4} \frac{1}{2} \frac{2}{1 + ze^{-ia}} - \frac{1}{1 + ze^{-ia}}
\]
\[
d \frac{1}{4} \frac{1}{2} \frac{2}{1 + ze^{-ia}} - \frac{1}{1 + ze^{-ia}}
\]
\[
d \frac{1}{4} \frac{1}{2} \frac{2}{1 + ze^{-ia}} - \frac{1}{1 + ze^{-ia}}
\]
\[
d \frac{1}{4} \frac{1}{2} \frac{2}{1 + ze^{-ia}} - \frac{1}{1 + ze^{-ia}}
\]
\[
d \frac{1}{4} \frac{1}{2} \frac{2}{1 + ze^{-ia}} - \frac{1}{1 + ze^{-ia}}
\]

We will show that \(A, B, C 2 D.\) Let

\[
\frac{1}{4} \frac{1}{2} \frac{2}{1 + ze^{-ia}} - \frac{1}{1 + ze^{-ia}}
\]
\[
\frac{1}{4} \frac{1}{2} \frac{2}{1 + ze^{-ia}} - \frac{1}{1 + ze^{-ia}}
\]
\[
\frac{1}{4} \frac{1}{2} \frac{2}{1 + ze^{-ia}} - \frac{1}{1 + ze^{-ia}}
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\frac{1}{4} \frac{1}{2} \frac{2}{1 + ze^{-ia}} - \frac{1}{1 + ze^{-ia}}
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We will show that \(A, B, C 2 D.\) Let

\[
\frac{1}{4} \frac{1}{2} \frac{2}{1 + ze^{-ia}} - \frac{1}{1 + ze^{-ia}}
\]
\[
\frac{1}{4} \frac{1}{2} \frac{2}{1 + ze^{-ia}} - \frac{1}{1 + ze^{-ia}}
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\frac{1}{4} \frac{1}{2} \frac{2}{1 + ze^{-ia}} - \frac{1}{1 + ze^{-ia}}
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\frac{1}{4} \frac{1}{2} \frac{2}{1 + ze^{-ia}} - \frac{1}{1 + ze^{-ia}}
\]

We will show that \(A, B, C 2 D.\) Let
where \( a \in \mathbb{C} \), \( n \in \mathbb{N} \). We apply Cohn’s rule to \( f(z) = z^3 + (\cos a + 2e^{-ib})z - 2e^{-ib} \). Note that \( jz^2 + 1/4 j \leq 1 \), thus we get
\[
\frac{1}{4} \left( \frac{1}{2} e^{-ib} \right) \cos a \frac{1}{2} e^{-ib} - \frac{1}{2} e^{-ib} \cos a \frac{1}{2} e^{-ib} : \]

Since \( \frac{1}{2} e^{-ib} \cos a \frac{1}{2} e^{ib} = \frac{1}{2} \cos a \frac{1}{2} e^{-ib} \leq \frac{1}{2} \cos a \frac{1}{2} e^{-ib} \leq \frac{1}{4} \cos a \frac{1}{4} \) (note that \( a \geq 1/4 \)), we can use Cohn’s rule again; this time on \( f_1 \).

We get
\[
f_1(z) = \frac{1}{4} \frac{1}{2} e^{-ib} \cos a \frac{1}{2} e^{ib} f_1(z) \]

Clearly \( f_2 \) has one zero at
\[
z = \frac{1}{4} \left( \cos a \frac{1}{2} e^{-ib} \right) \frac{1}{2} e^{-ib} \left( \cos a \frac{1}{2} e^{ib} \right)^2 - \frac{1}{4} \frac{1}{2} e^{-ib} \cos a \frac{1}{2} e^{ib} - \frac{1}{2} - \frac{1}{4} \cos a \frac{1}{4} \cos b \frac{1}{2} e^{-ib} : \]

If we put \( x = \cos a \), \( y = \cos b \), then \( x < -1, 0 \), \( y < -1, 1 \) and the above inequality becomes
\[
-\frac{3}{16} x^3 + 3 y x^2 + 6 x y - 6 x^3 y + 3 y^2 - 3 x y^2 + 3 y^2 x + 3 y^2 - 3 y^2 x - x^2 y^2 - y^2 x^2 + 2: 0: \]

Therefore, by Cohn’s rule, \( f \) has all its \( 3 \) zeros in \( D \), that is \( A, B, C \in D \) and so \( jzj < 1 \) for all \( z \in D \).

Next, consider the case in which \( f(z) = e^{ib} z^2 \). In this case,
\[
\frac{1}{4} - e^{ib} z^2 : \]

Hence, \( jzj < 1 \).

In proving the last theorem, we will use the following corollary of the Schur–Cohn algorithm.

**Corollary to the Schur–Cohn Algorithm** [10, p. 383] Given a polynomial
\[
f(z) = a_0 + a_1 z + \cdots + a_n z^n \]
of degree \( n \), let
\[
M = \frac{1}{4} \det \begin{pmatrix} A & B & Y \\ B & A & 1 \\ \end{pmatrix} = \frac{1}{4} \det \begin{pmatrix} a_n & \cdots & a_1 & a_0 \\ \end{pmatrix}, \]

where \( a_n, \ldots, a_0 \in \mathbb{C} \).
where \( A \transp \) is the conjugate transpose of \( A \), and \( A \) and \( B \) are the triangular matrices

\[
\begin{bmatrix}
0 & a_0 & a_1 & \cdots & a_{-1} & 0 \\
B & a_0 & a_1 & \cdots & a_{-2} & C \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
A & B & a_0 & a_1 & \cdots & a_{n-2} \\
0 & a_0 & a_1 & \cdots & a_{n-1} & 1
\end{bmatrix}
\]

\[
A \transp \begin{bmatrix}
0 & a_0 & a_1 & \cdots & a_{-1} & 0 \\
B & a_0 & a_1 & \cdots & a_{-2} & C \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
A & B & a_0 & a_1 & \cdots & a_{n-2} \\
0 & a_0 & a_1 & \cdots & a_{n-1} & 1
\end{bmatrix}
\]

The \( f \) has all of its zeros inside the unit circle if and only if the determinants \( M_1, \ldots, M_n \) are all positive.

**Theorem 6** Let \( f \in \mathbb{K}_0^m \) with \( h \circ \delta z \circ b \in \mathbb{K}_0^m \), \( \frac{1}{2} \frac{1}{1+i \sin a} \log \frac{1+bz}{1+bez^{-a}} \), where \( a \leq n \) and \( \delta z \in \mathbb{C} \). Then \( f \in \mathbb{S}_n^m \) and is convex in the direction of the real axis.

**Proof** Using Equation (8) with \( \delta z \) and simplifying, we have

\[
|z|^2 \begin{bmatrix}
1 & 3 & a & \cos abz^2 & 2a & \cos abz & \delta a & 2a & \cos abz & \delta a & \cos a & 1 & 0 & \frac{1}{2} & 0
\end{bmatrix}
\]

\[
1 \begin{bmatrix}
\frac{1}{2} & 0 & a & \cos abz & \delta a & 2a & \cos abz & \delta a & \cos a & 1 & 0 & \frac{1}{2} & 0
\end{bmatrix}
\]

\[
\frac{1}{4} - z \begin{bmatrix}
1 & 3 & a & \cos abz^2 & 2a & \cos abz & \delta a & 2a & \cos abz & \delta a & \cos a & 1 & 0 & \frac{1}{2} & 0
\end{bmatrix}
\]

\[
\frac{1}{4} - z \begin{bmatrix}
\delta z & A \delta z & B \delta z & C \delta z
\end{bmatrix}
\]

By the corollary to the Schur–Cohn algorithm, we need to show that the determinants \( M_1, M_2, M_3 \) are all positive (for convenience, let \( \cos a \in \frac{1}{4} \times \); so \(-1 \leq x \leq 0 \) and \(-1 \leq a \leq 1 \):)

\[
M_1 \begin{bmatrix}
1 & ax & \frac{1}{4} & 0
\end{bmatrix}
\]

\[
M_2 \begin{bmatrix}
1 & ax & \frac{1}{4} & 0
\end{bmatrix}
\]

\[
M_3 \begin{bmatrix}
1 & ax & \frac{1}{4} & 0
\end{bmatrix}
\]

where \( A \) is the conjugate transpose of \( A \), and \( A \) and \( B \) are the triangular matrices

\[
\begin{bmatrix}
0 & a_0 & a_1 & \cdots & a_{-1} & 0 \\
B & a_0 & a_1 & \cdots & a_{-2} & C \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
A & B & a_0 & a_1 & \cdots & a_{n-2} \\
0 & a_0 & a_1 & \cdots & a_{n-1} & 1
\end{bmatrix}
\]

The \( f \) has all of its zeros inside the unit circle if and only if the determinants \( M_1, \ldots, M_n \) are all positive.
Complex Variables and Elliptic Equations

since 0 < a < 1 and \(-1 < x < 0\). Assume a \(\frac{1}{4} a_0<0\) is fixed. Then, \(P(a_0, x)\) is increasing and attains its minimum at \(x = \frac{1}{4} - 1\). Thus,

\[
P(a_0, x) = \frac{1}{4} P(a_0, -1) = \left(\frac{1}{4} a_0 - 1\right)^2 = 0.
\]

Note, \(P(0, x) = 2 + x^4\).

\[
\begin{bmatrix}
0 & a_3 & 0 & 0 & a_0 & a_1 & a_2 \\
a_2 & a_3 & 0 & 0 & a_0 & a_1 & C \\
a_1 & a_2 & a_3 & 0 & a_0 & C & C \\
a_0 & 0 & a_3 & a_2 & a_1 & C & C \\
a_2 & a_1 & a_0 & 0 & 0 & a_3 & C \\
a_1 & a_0 & 0 & 0 & a_3 & a_2 & C \\
\end{bmatrix}
\]

\[
M_3 = \frac{1}{4} \det
\begin{bmatrix}
0 & 1 & 0 & 0 & ax & \frac{1}{2} a - \frac{1}{2} & a & 2ax & \frac{1}{2} b & a & x \\
\frac{1}{2} b & a & 1 & 0 & 0 & ax & \frac{1}{2} a - \frac{1}{2} & a & 2ax & C & C \\
\frac{1}{2} b & a & 2ax & 1 & 0 & 0 & ax & \frac{1}{2} a - \frac{1}{2} & a & 2ax & C \\
ax & \frac{1}{2} b & a - \frac{1}{2} & 0 & 0 & 1 & \frac{1}{2} b & a & x & a & C \\
ax & \frac{1}{2} b & 2ax & ax & \frac{1}{2} a - \frac{1}{2} & 0 & 0 & 1 & \frac{1}{2} b & a & x & A \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & a_3 & 0 & 0 & a_0 & a_1 & a_2 \\
a_2 & a_3 & 0 & 0 & a_0 & a_1 & C \\
a_1 & a_2 & a_3 & 0 & a_0 & C & C \\
a_0 & 0 & a_3 & a_2 & a_1 & C & C \\
a_2 & a_1 & a_0 & 0 & 0 & a_3 & C \\
a_1 & a_0 & 0 & 0 & a_3 & a_2 & C \\
\end{bmatrix}
\]

\[
\frac{1}{4} \frac{1}{2} b x + z^4 - b^3 b - ab b - 2ax - ab b b 3 a b = 0
\]

Therefore, \(A, B, C, D\) and \(j e(z)\) is 1 for all \(z \in D\).

Remark 2 Unlike Theorem 4, this result does not hold for \(-1 < a < -\frac{1}{3}\) since \(M_3 < 0\) for these values of \(a\).

Example 3 Let \(f_3 = h_3 + g_3\), where \(h_3, g_3 1 \log \left(1 - iz\right)\) (that is, \(a = \frac{1}{4} a\) in Theorem 5) with \(\frac{1}{4} z = z^2\). Then

\[
\begin{align*}
h_3(4) & = \frac{1}{4} \log \left(1 - iz\right) \frac{1}{4} b z - \frac{1}{4} b iz, \\
g_3(4) & = \frac{1}{4} \log \left(1 - iz\right) \frac{1}{4} b z - \frac{1}{4} b iz.
\end{align*}
\]

Consider \(F_3 = f_3 f_0\) \(G_3 f_3 \mathcal{G}_3\). From Equation (4) we derive

\[
\begin{align*}
H_3 H_0 & = \frac{1}{4} \log \left(1 - iz\right) \frac{1}{4} b z - \frac{1}{4} b iz, \\
G_3 G_0 & = \frac{1}{4} \log \left(1 - iz\right) \frac{1}{4} b z - \frac{1}{4} b iz.
\end{align*}
\]

From Equation (8), \(l \log \frac{1}{4} z^2\).

We now show that the image of the first quadrant of \(D\) under the mapping \(F_3\) is the domain whose boundary consists of the positive real axis, upper imaginary axis and the lines \(f_{\frac{1}{2}} b i y, y \in \frac{2}{3} b, f_{\frac{1}{2}} b i x, x \in \frac{2}{3} b\). We have

\[
\begin{align*}
F_3 & = \frac{1}{2} \frac{1}{4} \log \left(1 - iz\right) \frac{1}{4} b z - \frac{1}{4} b iz, \\
G & = \frac{1}{2} \frac{1}{4} \log \left(1 - iz\right) \frac{1}{4} b z - \frac{1}{4} b iz.
\end{align*}
\]
As in the previous two examples, we use the transformation \( \frac{j \pi}{1 + z} \). This transformation maps the part of the disc in the first quadrant onto the exterior of the unit disc contained in the first quadrant, and we note that the interval \([0, i)\) is mapped onto the quarter of the unit circle. If we put \( \frac{j \pi}{1 + z} \), \( r \geq 1 \), \( \beta \in [0, \pi/2) \), then we get

\[
\begin{align*}
\text{Re } F_3(z) &= \frac{1}{4} \arctan \frac{r - \frac{1}{r}}{2 \cos \frac{\beta}{2}} \cos \frac{\beta}{2} - \frac{1}{r} \cos \frac{\beta}{2} \\
\text{Im } F_3(z) &= \frac{1}{4} \frac{2 \sin \beta}{r - \frac{1}{r}} - \frac{1}{4} \frac{4 \cos^2 \beta}{\beta}.
\end{align*}
\]

One can see that the image of the quarter of the unit circle in the first quadrant in the \( j \)-plane under \( F_3 \) is the upper imaginary axis and the image of the line \( \beta = 1 \) is the positive real axis. Now we consider the level curves

\[
\beta = \frac{2 \sin 2\beta}{r - \frac{1}{r}} - \frac{1}{4} \frac{4 \cos^2 \beta}{\beta}, \quad c \leq 0:
\]

Since \( r \geq 1 \) and \( \beta \in (0, \pi/2) \), from above we get

\[
r = \frac{\tan \beta}{c - \frac{\beta}{2}} - 1:
\]

Let \( \beta_c \in (0, \pi/2) \) be the number satisfying the equation \( \tan \beta_c = c - \beta_c \). If \( 0 \leq c \leq \pi/2 \), we assume that \( \beta_c = \beta_c \), while if \( c \geq \pi/2 \), we assume that \( \beta_c \geq \beta_c \).
Using Equation (9) we find that on the level curve we have
\[
\text{Re } F_3 \left( \frac{1}{4} \right) = \frac{1}{4} \arctan \frac{\tan \frac{B}{c - B} - 1}{\cos^2 \frac{B}{c - B} - 1}.
\]

Using an analysis similar to the one in the previous examples, we get the result. The images of concentric circles inside D under the convolution map \( f_0 \ast \frac{1}{4} F_3 \) are shown in Figure 3.

References