Harmonic univalent mappings onto asymmetric vertical strips

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Abstract

Let \( \Omega_\alpha \) be the asymmetrical vertical strips defined by \( \Omega_\alpha = \{ w : \alpha - \frac{\pi}{2} < \text{Re} \ w < \alpha\} \), where \( \pi/2 \leq \alpha < \pi \), and let \( D \) be the unit disk. We characterize the class \( S_H(D, \Omega_\alpha) \) of univalent harmonic orientation-preserving functions \( f \) which map \( D \) onto \( \Omega_\alpha \) and are normalized by \( f(0) = 0 \), \( f'(0) = 0 \), and \( f_z(0) > 0 \). Then we use this characterization to demonstrate a few other results.

1 Introduction

Let \( S_H \) be the class of complex-valued harmonic functions \( f \) which are univalent and orientation-preserving mappings of the unit disk \( D = \{ z : |z| < 1 \} \) and are normalized by \( f(0) = 0 \) and \( f_z(0) = 1 \). Clunie and Sheil-Small [1] showed that functions in such a class have the form

\[
 f = h + g,
\]

where

\[
 h(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} b_k z^k
\]

are analytic in \( D \). They also showed that the orientation-preserving condition implies that \( |b_1| < 1 \) and so \( (f - b_1 f)/(1 - |b_1|^2) \in S_H \). Hence it is customary to just consider the subclass

\[
 S_H^0 = \{ f \in S_H \text{ with } f_z(0) = 0 \}.
\]

The uniqueness result of the Riemann Mapping Theorem does not extend to these classes of harmonic functions, and several authors have studied the subclass of functions that map \( D \) onto specific domains. In particular, Hengartner and Schober [3] considered the strip domain \( \Omega = \{ w : |\text{Im } w| < \pi/4 \} \). We will apply their results to derive a family
of functions that includes all mappings in $S^O_D$ from $D$ onto vertical strip domains that are asymmetric with respect to the imaginary axis. Using this, we will characterize all mappings in $S^O_D$ whose image is either a right-half plane or the entire plane minus a slit lying on the negative real axis.

2 Asymmetric vertical strip mappings

In [3], Hengartner and Schober investigated the family $S^H(D, \Omega)$ of normalized harmonic univalent mappings from the unit disk $D$ onto the horizontal strip $\Omega = \{ w : |\text{Im } w | < \pi/4 \}$. By the use of a rotation and a composition on their family of functions, we derive analogous results about the family of normalized univalent mappings from $D$ onto the vertical asymmetric strips.

In particular, let $f \in S^H(D, \Omega_\alpha)$, the family of normalized univalent mappings from $D$ onto the the vertical asymmetric strips $\Omega_\alpha = \{ w : \alpha - \pi/2 \sin \alpha < \text{Re } (w) < \alpha + \pi/2 \sin \alpha \}, \pi/2 \leq \alpha < \pi$. Recall that $f = h + g$, where $h, g$ are in the space of analytic functions, $H(D)$, on $D$, and that $|a(z)| = |g'(z)/h'(z)| < 1$. Now, $f = \text{Re } (h + g) + i \text{Im } (h - g)$. So

$$h(z) - g(z) = \int \frac{h'(z) - g'(z)}{h'(z) + g'(z)} [h'(z) + g'(z)]dz$$

$$= \int 1 - a(z) \cdot \frac{1}{1 + a(z)} \cdot \varphi'(z)dz,$$

where $\varphi(z) = h(z) + g(z)$.

Now $\varphi$ is the conformal map from $D$ onto $\Omega_\alpha$, normalized by $\varphi(0) = 0$ and $\varphi'(0) > 0$. To see this, note that if we consider the map $F(w) = \zeta = \xi + i\eta = \varphi(f^{-1}(w))$, then $f$ consists of the successive transformations $(u, v) \to (w, \overline{w}) \to (z, \overline{z}) \to (\varphi, \overline{\varphi}) \to (\xi, \eta)$ so that

$$\left( \frac{\partial \varphi}{\partial \overline{w}}, \frac{\partial \varphi}{\partial w} \right) = \left( \frac{1}{2}, -\frac{1}{2} \right) \left( \begin{array}{cc} h' + g' & 0 \\ 0 & \overline{h'} + \overline{g'} \end{array} \right) \left( \begin{array}{cc} \frac{h'}{\Delta} & -\frac{g'}{\Delta} \\ -\frac{\overline{g'}}{\Delta} & \frac{\overline{h'}}{\Delta} \end{array} \right) \left( \begin{array}{c} 1 \\ i \end{array} \right)$$

and thus $\frac{\partial \varphi}{\partial \overline{w}} = 1, \frac{\partial \varphi}{\partial w} = 0$ and $\frac{\partial h}{\partial \overline{w}} = \frac{|h'|^2 + |g'|^2}{\Delta}$, where $\Delta = |h'|^2 - |g'|^2 = \text{Re } [(h' + g')(h' - g')]$. Therefore, $\varphi$ is a univalent map from $D$ onto a vertical strip. Because of the normalization of $\varphi$, we see that $\varphi$ is the map $(1/2) \log[(1+z)/(1-z)]$ rotated by $-iz$, composed with the Möbius transformation $(z + p)/(1 + pz)$, where $0 < p < 1$, and normalized.
Hence, any map \( f \) in \( S_H(D, \Omega_\alpha) \) is of the form

\[
  f(z) = \text{Re} \varphi(z) + i \text{Im} \int \frac{1 - a(z)}{1 + a(z)} \cdot \varphi'(z)dz.
\]

\[
  = \varphi(z) - 2i \int \frac{a(z)}{1 + a(z)} \cdot \varphi'(z)dz.
\]

(1)

Since \( a \) is in \( H(D) \), \( |a(z)| < 1 \) on \( D \), and \( a(0) = 0 \), we have

\[
  \frac{1 - a(z)}{1 + a(z)} = \int_{|\eta| = 1} \frac{1 + \eta z}{1 - \eta z} d\mu(\eta),
\]

where \( \mathcal{P} \) is the set of probability measures on the Borel sets of \( |\eta| = 1 \).

**Definition 2.1.** For \( z \in D \) and \( |\eta| = 1 \), define the kernel

\[
  K(z, \eta) = \int_0^z \frac{1 + \eta w}{1 - \eta w} \frac{1}{(1 + we^{i\alpha})(1 + we^{-i\alpha})} dw
\]

\[
  = \begin{cases} 
  \frac{\cos \alpha}{2 \sin^2 \alpha} \log \left( \frac{1 + ze^{i\alpha}}{1 + ze^{-i\alpha}} \right) \left( 1 + \eta e^{i\alpha} \right) 
  + \frac{1}{i \sin \alpha} \left( \frac{ze^{i\alpha}}{1 + ze^{i\alpha}} \right), & \text{if } \eta = -e^{i\alpha} \\
  \frac{\cos \alpha}{2 \sin^2 \alpha} \log \left( \frac{1 + ze^{-i\alpha}}{1 + ze^{i\alpha}} \right) 
  - \frac{1}{i \sin \alpha} \left( \frac{ze^{-i\alpha}}{1 + ze^{i\alpha}} \right), & \text{if } \eta = -e^{-i\alpha} \\
  \frac{1}{2i \sin \alpha} \left( \frac{1 - \eta e^{i\alpha}}{1 + \eta e^{i\alpha}} \right) \log \left( \frac{1 - \eta}{1 + \eta e^{i\alpha}} \right) 
  - \frac{1}{2i \sin \alpha} \left( \frac{1 - \eta e^{-i\alpha}}{1 + \eta e^{-i\alpha}} \right) \log \left( \frac{1 - \eta}{1 + \eta e^{-i\alpha}} \right), & \text{if } \eta \neq -e^{\pm i\alpha} 
  \end{cases}
\]

Define the family

\[
  \mathcal{F}_\alpha = \{ f : f(z) = \text{Re} \left[ \frac{1}{2i \sin \alpha} \log \left( \frac{1 + ze^{i\alpha}}{1 + ze^{-i\alpha}} \right) \right] 
  + i \text{Im} \int_{|\eta| = 1} K(z, \eta) d\mu(\eta), \mu \in \mathcal{P} \}
\]

where \( \mathcal{P} \) is the set of probability measures on the Borel sets of \( |\eta| = 1 \).

From our discussion above, we obtain an isomorphism between the family \( S_H(D, \Omega) \) from Hengartner and Schober [3] and the class \( S_H(D, \Omega_\alpha) \). Hence we have the following theorem.
Theorem 2.2. The following properties hold:

1. If $f$ is a univalent harmonic and orientation preserving map from the unit disk $D$ onto $\Omega_\alpha = \{w: \frac{\alpha - \pi}{2\sin\alpha} < \text{Re}(w) < \frac{\alpha}{2\sin\alpha}\}$ such that $f(0) = 0$ and $f_z(0) > 0$, then $f_z(0) = 1$.

2. The set $S(H, D, \Omega) \subset F_{\alpha}^*$ with $S(H, D, \Omega) = F_{\alpha}$.

3. If $f \in F_{\alpha}$, then $f(D)$ is either the strip $\Omega_{\alpha}$, a halfstrip, a triangle, or a trapezium.

3 Consequences

The results from the previous section yield a few nice consequences.

Theorem 3.1. Every right-half plane mapping $f \in S^0_H$ can be expressed as a limit of functions in $F_{\alpha}$. In particular, $f$ maps $\partial D$ into the line $\text{Re} w = -\frac{1}{2}$.

Proof. This follows from the normality of the family $S^0_H$ and an approximation theorem (theorem 3.7 in [1]).

Corollary 3.2. Let $f = h + g \in S^0_H$ be a right-half plane mapping. Then

$$f(z) = h(z) + g(z) - 2i \text{Im} g(z) = \frac{z}{1-z} - 2i \text{Im} \int_0^{2\pi} K(z, t) d\mu(t),$$

where

$$K(z, \eta) = \begin{cases} 
-\frac{1}{2}z^2/(1-z)^2 & \text{if } \eta = 1 \\
\frac{z}{(1-\eta)(1-z)} + \frac{1}{(1-\eta)(1-\eta)} \log \left( \frac{1-z}{1-\eta z} \right) & \text{if } \eta \neq 1
\end{cases}$$

Proof. Let $f \in S(H, D, \Omega)$, where $f$ is of the form in (1). The result follows from taking the limit of $f$ as $\alpha \to \pi$.

Corollary 3.2 provides a general description for right-half plane mappings in $S^0_H$, so that in such cases we know that $h(z) + g(z) = z/(1-z)$. In a similar fashion, it has been shown that all slit mappings in $S^0_H$ whose slit lie on the negative real axis have the property that $h(z) - g(z) = 1/(1-z)^2$ ([2] or see [4]). Corollary 3.3 provides another proof of this.
Corollary 3.3. Let \( f = h + \overline{g} \in S^\circ_H \) be a slit mapping whose slit lies on the negative real axis. Then

\[
h(z) - g(z) = \frac{z}{(1 - z)^2}.
\]

Proof. Sheil-Small (Remark 7 in [5]) showed that if \( f = h + \overline{g} \in S^\circ_H \) is starlike, then \( \hat{f} = \hat{h} - \hat{g} \) is convex in \( S^\circ_H \), where

\[
\hat{h}(z) = \int_0^z \frac{h(w)}{w} \, dw \quad \text{and} \quad \hat{g}(z) = \int_0^z \frac{g(w)}{w} \, dw.
\]

Let \( f = h + \overline{g} \in S^\circ_H \) be a slit mapping whose slit lies on the negative real axis. Then \( \hat{f} \) is convex. In particular, \( \hat{f} \) is a right-half plane mapping since the process \( \hat{f}(z) = \int_0^z f(w)/w \, dw \) makes the boundary of \( \hat{f} \) normal to the boundary of \( f \). Hence, by Corollary (3.2)

\[
\frac{z}{1-z} = \hat{h}(z) - \hat{g}(z) = \int_0^z \frac{h(w) - g(w)}{w} \, dw.
\]

Therefore,

\[
\frac{z}{(1 - z)^2} = h(z) - g(z).
\]

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References