Math 313 Midterm III KEY
Winter 2010
sections 003 and 005
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Honor Code: After I have learned of the contents of this exam by any means, I will not disclose to anyone any of these contents by any means until after the exam has closed:

Signature:
1) Find a basis for the nullspace of $A$:

$$A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}. \quad (1)$$

What is the nullity of $A$?

**15pts**

**Solution**

$$\text{Nul} \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix} = \text{Nul} \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & -14 & -14 & -14 & -28 \\ 0 & 4 & 4 & 4 & 8 \\ 0 & -5 & -5 & -5 & -10 \end{bmatrix}$$

$$= \text{Nul} \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \text{Nul} \begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -x_3 - 2x_4 - x_5 \\ -x_3 - x_4 - 2x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad x_3, x_4, x_5 \in \mathbb{R}$$

$$= \begin{bmatrix} x_3 & 1 + x_4 & 0 + x_5 \end{bmatrix}, \quad x_3, x_4, x_5 \in \mathbb{R}$$

$$\text{Span} \begin{bmatrix} -1 & -2 & -1 \\ -1 & -1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{Span} S. \quad (2)$$

$S$ is a basis for the nullspace, perhaps the most natural one in some respects. Since this vector space is evidently 3 dimensional, the nullity of the matrix $A$ is 3.

2) Determine bases for both the row and column spaces of the matrix $A$ of problem 1). What is the rank of $A$? For all parts of this question, make sure you write something that is correct regardless of the matrix, i.e. indicate an explanation of
why your answer is correct. (There are accidental ways to get parts of this problem right; I am hoping to eliminate accidents.)

15pts

Solution

In the solution of problem 1) we see that the matrix $A$ is row-equivalent to the matrix

$$
\begin{pmatrix}
1 & 0 & 1 & 2 & 1 \\
0 & 1 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

(3)

So since row-equivalent matrices have the same row space, it follows that

$$
\text{Row } A = \text{Span } \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix} \right\},
$$

(4)

which indicates a basis for the row space. And since the first two columns of the matrix in (3) are clearly a basis for its column space, then, theorem, the first two columns of $A$ are a basis for the column space of $A$:

$$
\text{Col } A = \text{Span } \left\{ \begin{pmatrix} 1 \\ 3 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 0 \\ 3 \end{pmatrix} \right\}.
$$

(5)

Since $\text{Range } A = \text{Col } A$ is evidently 2-dimensional, the rank of $A$ is 2. (When we write $\text{Range } A = \text{Col } A$, we are thinking of $A$ as a linear mapping—left multiplication by $A$ is the relevant linear mapping. Of course, theorem, we also have $\dim \text{Col } A = \dim \text{Row } A$, so that $\text{rank } A = \dim \text{Range } A = \dim \text{Col } A = \dim \text{Row } A$.)

3) Let $W = \text{Row } A$, $A$ the matrix of problem 1). $W$ is a subspace of the vector space $V = \mathbb{R}^5$. Find a basis for the orthogonal complement $W^\perp$ of $W$.

15pts
Solution

Since, theorem, RowA and NulA are orthogonal complements, we have

\[ W^\perp = (\text{Row}A)^\perp = \text{Nul}A = \text{Span} \begin{bmatrix} 1 & -2 & -1 \\ -1 & -1 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} =: \text{Span} S. \]  

(6)

So \( S \) is a basis for \( W^\perp = (\text{Row}A)^\perp \). (From (4), one certainly sees that this space is orthogonal to \( \text{Row}A \)—and vice versa—and that, as per theorem, the dimensions of these spaces add up to that of the ambient space \( V = \mathbb{R}^5 = \text{Dom}A \):

\[ \text{Rank}A = \dim \text{Row}A + \text{Nullity}A = \dim \text{Nul}A = \# \text{ of columns of } A \left( = \dim \text{Dom}A \right) \]

4) Using the Gram-Schmidt process, find an orthogonal basis for \( \text{Col}A \), \( A \) the matrix of problem 1). Don’t worry about finding an orthonormal basis—I don’t care whether or not you normalize. Also, you might consider eliminating fractions to make life easier, as per the discussion in class.

5 points here are given for CHECKING YOUR ANSWER: if you find your answer is incorrect, but state this, you will at least get 5 points (if you are correct in this assessment of error).

15pts

Solution

Since by (5)

\[ \text{Col}A = \text{Span} \begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \end{bmatrix} =: \text{Span} \begin{bmatrix} u_1, u_2 \end{bmatrix} \]  

(7)

we can take the basis to be \( S = \{v_1, v_2\} \) where \( v_1 = u_1 \) and then

\[ v_2 \propto u_2 - \frac{v_1}{\langle v_1, v_1 \rangle} v_1 = u_2 - \frac{u_1}{\langle u_1, u_1 \rangle} u_1, \]

(8)
for then
\[ \langle v_1, v_2 \rangle \propto \left( u_1, u_2 - \frac{\langle u_1, u_2 \rangle}{\langle u_1, u_1 \rangle} u_1 \right) = \langle u_1, u_2 \rangle - \frac{\langle u_1, u_2 \rangle}{\langle u_1, u_1 \rangle} \langle u_1, u_1 \rangle = 0 \]  \hspace{1cm} (9)

as desired for any inner product \( \langle , , \rangle \), and we get \( v_2 = 0 \) from (8) (with nonzero proportionality indicated in (8) by \( \propto \)) iff

\[ u_2 = \frac{\langle u_1, u_2 \rangle}{\langle u_1, u_1 \rangle} u_1 \in \text{Span} \{ u_1 \}, \]  \hspace{1cm} (10)

i.e. iff \( \{ u_1, u_2 \} \) wasn’t a basis in the first place. (Here I am reminding of why Gram-Schmidt works, at least in the case of a 2-dimensional subspace.) In any event, for the case at hand, we have (8) gives

\[ v_2 \propto u_2 - \frac{\langle u_1, u_2 \rangle}{\langle u_1, u_1 \rangle} u_1 = u_2 - \frac{u_1 \cdot u_2}{u_1 \cdot u_1} u_1 = \begin{pmatrix} 4 \\ -2 \\ 0 \\ 3 \end{pmatrix} - \frac{4}{15} \begin{pmatrix} 1 \\ 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 0 \\ 3 \end{pmatrix} - \frac{4}{15} \begin{pmatrix} 1 \\ 3 \\ -1 \\ 2 \end{pmatrix} \approx \begin{pmatrix} 1 \\ 3 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 56 \\ -42 \\ 4 \\ 37 \end{pmatrix} =: v_2, \]  \hspace{1cm} (11)

so that an orthogonal basis is

\[ S = \{ v_1, v_2 \} = \begin{pmatrix} 1 \\ 3 \\ -1 \\ 2 \end{pmatrix}, \]  \hspace{1cm} (12)

One could check that

\[ v_1 \cdot v_2 = 56 - 126 - 4 + 74 = 0 \]  \hspace{1cm} (13)

and that
which confirms that each element of the new basis is in the span of the old.

5) Verify that

\[ S = \{v_1, v_2\} = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix} \]  

is an orthogonal basis for a certain 2-dimensional subspace of \( \mathbb{R}^4 \) (namely \( \text{Span} \, S \)). Using this fact, find the coordinate vector \( (w)_S \) of the vector \( w \in \text{Span} \, S \), where

\[
w = \begin{pmatrix} 1 \\ 5 \\ -9 \\ 10 \end{pmatrix}
\]

15pts

Solution

The basis is orthogonal since the Euclidean inner product of the two vectors is

\[ 1 + 6 - 3 - 4 = 0 \]  

(And the set is clearly linearly independent by all accounts, hence a basis for its span.) By theorem then we have

\[
w = \frac{\langle v_1, w \rangle}{\langle v_1, v_1 \rangle} v_1 + \frac{\langle v_2, w \rangle}{\langle v_2, v_2 \rangle} v_2
\]

so that
\[ (w)_{w} = \left( \frac{\langle v_1, w \rangle}{\langle v_1, v_1 \rangle}, \frac{\langle v_2, w \rangle}{\langle v_2, v_2 \rangle} \right)^T = \left( \frac{v_1 \cdot w}{v_1 \cdot v_1}, \frac{v_2 \cdot w}{v_2 \cdot v_2} \right)^T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 5 & 2 & 5 \\ -1 & -9 & -2 & -9 \\ 2 & 10 & -10 & 10 \end{pmatrix}^T = \begin{pmatrix} 1 \\ 3 \\ 3 \\ 1 \\ 1 \\ 2 \\ 2 \\ 3 \\ 3 \\ -1 \\ -1 \\ -2 \\ -2 \end{pmatrix} \]

\[
\begin{pmatrix}
1+15+9+20 \\
1+10-27-20 \\
1+9+1+4 \\
1+4+9+4
\end{pmatrix}^T = \begin{pmatrix} 45 \\ -36 \\ 15 \\ 18 \end{pmatrix} = (3, -2)^T.
\]

6) Find the least squares solution \( x \) to the inconsistent system \( Ax = b \), where

\[ A = \begin{bmatrix}
1 & 1 \\
3 & 2 \\
-1 & 3 \\
2 & -2
\end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 5 \\ -8 \\ 11 \end{bmatrix} \]

15 pts

**Solution**

The least squares system \( A^T A x = A^T b \) is gotten by noting that

\[
A^T A = \begin{bmatrix}
1 & 3 & -1 & 2 \\
3 & 2 & -1 & 3 \\
2 & -2
\end{bmatrix}, \quad A^T b = \begin{bmatrix} 0 \\ 5 \\ -8 \\ 11 \end{bmatrix}
\]

\[
A^T A = \begin{bmatrix}
1+9+1+4 & 1+6-3-4 \\
1+6-3-4 & 1+4+9+4
\end{bmatrix} = \begin{bmatrix} 15 & 0 \\ 0 & 24 \end{bmatrix},
\]

\[
A^T b = \begin{bmatrix} 15+8+22 \\ 10-24-22 \end{bmatrix} = \begin{bmatrix} 45 \\ -36 \end{bmatrix},
\]

so that clearly the least squares solution is given by
7) Find the transition matrix $P_{B'B}$ from bases $B$ to basis $B'$, where

$$B = \begin{pmatrix} 1 & 3 & 2 \\ 56 & -42 & 37 \end{pmatrix}, \quad B' = \begin{pmatrix} 1 & 3 \\ 4 & -1 \\ 37 & 2 \end{pmatrix}.$$  \tag{22}$$

15pts

Solution

By definition of transition matrix we require

$$(v)_{B'} = P_{B'B} (v)_B,$$ \tag{23}$$

where the $v$’s with subscripts are coordinate vectors with respect to the indicated bases. Exploiting the way in which matrix multiplication proceeds, by definition of coordinate vector we require that for any $v \in \text{Span}B' = \text{Span}B$

$$v = B'(v)_{B'} = B(v)_B,$$ \tag{24}$$

where now the $B$’s indicate matrices whose columns are the basis elements of the original $B$’s. So clearly then we obtain $P_{B'B}$ by putting in reduced row-echelon form the augmented matrix $\left[ \begin{array}{c|c} B' & B \end{array} \right]$, which proceeds as in (14), and which then gives

$$\begin{pmatrix} 1 & 4 & 1 & 56 \\ 3 & -2 & 3 & -42 \\ -1 & 0 & -1 & 4 \\ 2 & 3 & 2 & 37 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 1 & 56 \\ 0 & -14 & 0 & -210 \\ 0 & 4 & 0 & 60 \\ 0 & -5 & 0 & -75 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 1 & 56 \\ 0 & 10 & 15 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -4 \\ 0 & 10 & 15 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ \tag{25}$$
so that

\[ P_{BB} = \begin{bmatrix} 1 & -4 \\ 0 & 15 \end{bmatrix}. \]  

(26)

Indeed one confirms that if

\[ (v)_B = \begin{pmatrix} x \\ y \end{pmatrix} \]  

(27)

so that

\[
\begin{bmatrix} 1 \\ 3 \\ 2 \\ -1 \\ 4 \\ 37 \end{bmatrix} x + \begin{bmatrix} 56 \\ -42 \\ 4 \\ -1 \\ 4 \\ 37 \end{bmatrix} y = \begin{bmatrix} 1 \\ 56 \\ x \\ -42 \\ y \end{bmatrix} \]

so that

\[
\begin{bmatrix} 1 & 4 \\ 3 & -2 \\ -1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 & -2 \\ -1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \]

(28)

then we also have

\[
\begin{bmatrix} 1 & 4 \\ 3 & -2 \\ -1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 & -2 \\ -1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \]

(29)

as required.

8) What condition must \( a \) and \( b \) satisfy so that the matrix

\[
A = \begin{bmatrix} a+b & b-a \\ a-b & b+a \end{bmatrix} \]  

is orthogonal.

**15pts**

**Solution**

By definition a square matrix \( A \) is orthogonal iff \( A^T A = I \), \( I \) being the (relevant) identity. So the necessary and sufficient condition with \( A \) of the form in (30) is gotten by demanding that
\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} = I = A^T A = \begin{bmatrix}
  a+b & a-b \\
  b-a & b+a
\end{bmatrix} = \begin{bmatrix}
  (a+b)^2 + (a-b)^2 & b^2 - a^2 - b^2 + a^2 \\
  b^2 - a^2 - b^2 + a^2 & (a-b)^2 + (a+b)^2
\end{bmatrix}
\]
\[
= 2 \begin{bmatrix}
  a^2 + b^2 & 0 \\
  0 & a^2 + b^2
\end{bmatrix},
\]
(31)

and the condition is evidently that
\[
a^2 + b^2 = \frac{1}{2}.
\]
(32)

9) \( \langle \cdot, \cdot \rangle \) defined by
\[
\langle p, q \rangle = \int_0^1 p(x)q(x)\,dx
\]
(33)
is an inner product on \( P_2 = \text{Span}S = \text{Span} \{1, x, x^2\} = \text{Span} \{u_1, u_2, u_3\} \), where \( S \) is the standard basis for \( P_2 \). Find a basis for \( P_2 \) that is orthogonal with respect to \( \langle \cdot, \cdot \rangle \) by applying Gram-Schmidt to the standard basis. Please avoid fractions.

15pts

Solution

We get an orthogonal basis \( S' = \{v_1, v_2, v_3\} \) by choosing
\( \mathbf{v}_1 = \mathbf{u}_1 = 1, \)

\[ \mathbf{v}_2 \propto \mathbf{u}_2 - \left( \frac{\langle \mathbf{v}_1, \mathbf{u}_2 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \right) \mathbf{v}_1 = \int_0^1 x \, dx = \frac{1}{2} \]

\( \propto 2x-1 =: \mathbf{v}_2, \)

\[ \mathbf{v}_3 \propto \mathbf{u}_3 - \left( \frac{\langle \mathbf{v}_1, \mathbf{u}_3 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \right) \mathbf{v}_1 - \left( \frac{\langle \mathbf{v}_2, \mathbf{u}_3 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \right) \mathbf{v}_2 = \int_0^1 x^2 \, dx = \frac{1}{3} - \frac{2}{3} = \frac{1}{3} \]

\[ = x^2 - \frac{1}{3} \cdot \frac{1}{2} (2x-1) = 6x^2 - \frac{6}{3} - 3(2x-1) = 6x^2 - 6x + 1 =: \mathbf{v}_3. \quad (34) \]

10) Find the linear function \( q(x) \) closest to the quadratic polynomial
\[ p(x) = 6x^2 - 3x + 3, \]
where the notion of distance is given (indirectly) by the inner product in (33), i.e. where the notion of distance is given by
\[ d^2(p, q) = \int_0^1 (p(x) - q(x))^2 \, dx. \quad (35) \]

15pts

Solution

Since \( p(x) = 6x^2 - 3x + 3 = 3x + 2 + 1 \cdot (6x^2 - 6x + 1), \) and since
\[ \mathbf{v}_3 = 6x^2 - 6x + 1 \in \left( \text{Span } \{1, x\} \right)^\perp = \left( \text{Span } \{1, 2x - 1\} \right)^\perp, \] the answer is, theorem,
\[ q(x) = 3x + 2. \quad (36) \]

Alternatively, we have, theorem (from the orthogonal basis of the previous problem)
\[ q(x) = \frac{\langle v_1, p \rangle}{\langle v_1, v_1 \rangle} v_1(x) + \frac{\langle v_2, p \rangle}{\langle v_2, v_2 \rangle} v_2(x) = \frac{\langle 1, 6x^2 - 3x + 3 \rangle}{1} + \frac{\langle 2x - 1, 6x^2 - 3x + 3 \rangle}{\frac{1}{3}} (2x - 1) \]

\[ = \langle 1, 6x^2 - 3x + 3 \rangle + 3 \langle 2x - 1, 6x^2 - 3x + 3 \rangle (2x - 1) \]

\[ = \int_{0}^{1} (6x^2 - 3x + 3) \, dx + 3 \int_{0}^{1} (2x - 1)(6x^2 - 3x + 3) \, dx (2x - 1) \]

\[ = 2 - \frac{3}{2} + 3 \int_{0}^{1} (2x - 1)(6x^2 - 3x + 3) \]

\[ = 5 - \frac{3}{2} + 3 \int_{0}^{1} (6x^2 - 3x + 3 + 12x - 3 - 6x^2 + 6x) \, dx (2x - 1) \]

\[ = 5 - \frac{3}{2} + 3 \int_{0}^{1} (12x^3 - 12x^2 + 9x - 3) \, dx (2x - 1) = 5 - \frac{3}{2} + 3 \left( -4 + \frac{9}{2} - 3 \right) (2x - 1) \]

\[ = 5 - \frac{3}{2} + 3 \left( -4 + \frac{9}{2} \right) (2x - 1) \]

\[ = 5 - \frac{3}{2} + \frac{3}{2} (2x - 1) = 5 - \frac{3}{2} - \frac{3}{2} + 3x = 5 - 3 + 3x = 2 + 3x, \quad (37) \]

as advertised.