Section A

1. (5 points) Let $a$ and $b$ be positive numbers with $a > b$. Solve the initial value problem

$$y' + ay = e^{-bx}, \quad y(0) = 0.$$ 

Does the solution tend to a constant as $x \to \infty$?

2. (5 points) Find an integrating factor of $(1 - t^2)y' = ty + 1 + t$. Do not solve the equation.

3. (5 points) Find the general solution of the differential equation $y' = \sin(x)(y^2 - 1)$

4. (5 points) Sketch the integral curves of

$$\frac{dy}{dt} = 3y(9 - 3y)(2y - 12)$$

for the initial conditions $y(0) = 1, y(0) = 4$ and $y(0) = 6$. Do not solve the differential equation. Find all the equilibrium solutions and indicate whether they are stable or unstable.

5. (5 points) Suppose that $y_1(x)$ and $y_2(x)$ are two solution of the differential equation

$$y'' + \frac{2}{1 + x}y' + q(x)y = 0,$$

with $q(x)$ a continuous function; and suppose further that the Wronskian, $W(y_1, y_2)$ is not zero at the point $x = 0$. Show that the the Wronskian is never zero. Are $y_1(x), y_2(x)$ linearly dependent?

6. (5 points) Explain why the initial value problem

$$y' = 1 - y^{2/3}, \quad y(0) = 1,$$

has a unique solution in certain interval containing the point $t = 0$.

7. (5 points) Let $y_1(t)$ be a solution of the initial value problem

$$y'' + p(t)y' + q(t)y = 0, \quad y(0) = a, \quad y'(0) = b,$$

and let $y_2(t)$ be a solution of the initial value problem

$$y'' + p(t)y' + q(t)y = f(t), \quad y(0) = 0, \quad y'(0) = 0.$$ 

Show that $y(t) = y_1(t) + y_2(t)$ is a solution of

$$y'' + p(t)y' + q(t)y = f(t), \quad y(0) = a, \quad y'(0) = b.$$
Section B

1. (15 points) Determine if the ODE

\[(e^{2y} - y \cos(xy))dx + (2xe^{2y} - x \cos(xy) + 2y)dy = 0\]

is exact and if so, find the solution when the initial condition is given by \(y(0) = 3\).

2. Find the general solutions for the following linear ODE’s

a) (8 points)

\[y'' - 10y' + 25y = 10e^{5t} + \cos(5t)\]

b) (7 points)

\[y''' + 9y'' + 24y' + 16y = 0\]

3.

a) (5 points) Determine an interval in which the solution of the following initial value problem is certain to exist (justify your answer).

\[(t - 1)y'' - ty' + y = 0 \quad (1)\]
\[y(-2) = 3, \quad y'(-2) = -1 \quad (2)\]

b) (10 points) A particular solution for equation (1) is given by \(y_1(t) = e^t\). Find a second solution linearly independent with \(y_1(t)\)

4. When a man bought his house he needed to borrow from the bank $185,000. The annual interest rate at that time was 8.5% and the mortgage term 30 years. Assuming that interest is compounded continuously and that payments are also made continuously at a constant rate, answer the following questions:

a) (3 points) Write down an initial value problem satisfied by \(y(t)\), the mortgage at time \(t\).

b) (3 points) How much does this buyer pay after one year?

c) (8 points) If after the first year he refinances his mortgage at the interest rate of 6.5% for the next 29 years. Write down an initial value problem satisfied by \(y(t)\), the mortgage at time \(t \geq 1\).

d) (3 points) What is his new monthly payment?

e) (3 points) How much more over the 30 years would he be paying if he had not refinanced?