Math 511 Fall 2001 Project III
Due: Nov 14, 2001

Consider the variable coefficient linear elliptic problem:

\[
x^2 \frac{\partial^2 u}{\partial x^2} + \frac{6}{\pi^2} \frac{\partial^2 u}{\partial y^2} = -6 \cos(\pi y) - 3,
\]

over the unit square \([0,1] \times [0,1]\), with boundary conditions

\begin{align*}
(N) \quad u(x, 1) &= -1 - \pi^2 - x^3 \\
(E) \quad u(1, y) &= 2 \cos(\pi y) - \frac{1}{4} \pi^2 (y + 1)^2 \\
(S) \quad u(x, 0) &= 1 - \frac{1}{4} \pi^2 + x^3 \\
(W) \quad u(0, y) &= \cos(\pi y) - \frac{1}{4} \pi^2 (y + 1)^2
\end{align*}

The exact solution of this problem is

\[
u(x, y) = (1 + x^3) \cos(\pi y) - \frac{\pi^2}{4} (y + 1)^2
\]

Discretize with 17 grid lines in each direction. Use the alternating Schwarz method to find an approximation solution and compare it with the exact solution. You should terminate the iteration when the values at each grid point between successive iterations do not vary by more than \(0.5 \times 10^{-4}\) or when the number of iterations become excessive.

1. Decompose the unit square into two non-overlapping sub-domains along the line \(x = \frac{1}{2}\). Assuming initially a zero solution value at the interior points along \(x = \frac{1}{2}\), use an alternating Dirichlet-Neumann boundary condition strategy in your computation to find an approximate solution.

2. Decompose the unit square into two overlapping sub-domains, with the line \(x = \frac{1}{2}\) being contained in the interior of one subdomain and lying on the boundary of the other subdomain. Assuming initially a zero solution value at the interior points along \(x = \frac{1}{2}\), use an alternating Dirichlet-Dirichlet boundary condition strategy in your computation to find an approximate solution.

3. Compare the result of the overlapping and the non-overlapping approaches. Which method would you prefer? Why?

4. [Optional] Repeat the numerical study with 33 and 65 grid lines.