Section A- attempt at most 5 questions

1. Does the floating point system possess the associativity property, i.e. \( a + (b + c) = (a + b) + c \)? Explain. Describe the term “catastrophic cancellation” and how it can be avoided. [5]

2. Define the term “condition number” of a matrix \( A \). Explain the significance of the condition number in numerical solution of the system \( Ax = b \). What is the Cholesky factorization of a symmetric positive definite matrix? [5]

3. Use the bisection to find a zero of the function \( f(x) = x - \tan(x) \). You are only required to perform three iterations. Explain you choice of initial interval. Since \( f(x) \) is clearly discontinuous, and the bisection method only works when the function is continuous, does it still lead to a zero of the given function? why? [5]

4. Compute the least squares polynomial of the form \( y = bx^a \) for the following data [5]

\[
\begin{array}{c|c|c}
 x & 4.0 & 4.5 & 5.1 \\
y & 102.56 & 130.11 & 167.53 \\
\end{array}
\]

5. By considering the function and the first and second derivative values at \( x = 1 \), determine the coefficients \( a, b, c, d \) so that the function

\[
S(x) = \begin{cases} 
3 + x - 9x^2, & x \in [0, 1] \\
 a + b(x-1) + c(x-1)^2 + d(x-1)^3, & x \in [1, 2] 
\end{cases}
\]

is a clamped cubic spline for some \( f \) with \( f'(2) = 5 \). [5]

6. Use the three point Gauss rule

\[
\int_{-1}^{1} f(x)dx = \frac{5}{9} f(-\sqrt{\frac{3}{5}}) + \frac{8}{9} f(0) + \frac{5}{9} f(\sqrt{\frac{3}{5}})
\]

to find an approximate value for the following integral [5]

\[
\int_{1}^{1.6} \frac{2x}{x^2 - 4}dx.
\]
Section B- attempt all three questions

I Given the following data

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

(a) Find the Newton divided difference polynomial. [4]
(b) Find the Lagrange interpolating polynomial. (You must clearly define the basis Lagrange polynomials $L_{n,k}(x)$.) [4]
(c) Is it true that the polynomials obtained about are not the same? Explain. [1]
(d) Is it desirable to approximate a function with a polynomial that interpolates the function at 1000 evenly spaced points? Why? [1]

II (a) Derive the central difference formula

$$f'(x) = \frac{f(x + h) - f(x - h)}{2h} + O(h^2)$$

and explain why in practice an accurate derivative value cannot be obtained by making $h$ extremely small. [3]
(b) Use extrapolation to derive an $O(h^4)$ formula for $f'(x)$. [3]
(c) Fill in the missing entries in the following Romberg table for the integral $\int_1^2 e^{-x}dx$ [4]

<table>
<thead>
<tr>
<th>$h$</th>
<th>$R_{i,1}$</th>
<th>$R_{i,2}$</th>
<th>$R_{i,3}$</th>
<th>$R_{i,4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.251667</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>?</td>
<td>?</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{4}$</td>
<td>0.23375407</td>
<td>0.23254917</td>
<td>0.23254427</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{8}$</td>
<td>0.23284687</td>
<td>0.23254447</td>
<td>?</td>
<td>0.23254416</td>
</tr>
</tbody>
</table>

III (a) We wish to derive the Simpson’s rule with error term from the formula

$$\int_{x_0}^{x_2} f(x) = a_0 f(x_0) + a_1 f(x_1) + a_2 f(x_2) + k f^{(4)}(\xi).$$

i. How are the integration points $x_0, x_1$ and $x_2$ related? [1]
ii. Determine the coefficients $a_0, a_1$ and $a_2$ using the fact that the Simpson’s rule is exact for cubic polynomials. [4]
iii. Find $k$ by applying the integration formula with $f(x) = x^4$. [2]
(b) Apply the composite Simpson’s rule for 8 subintervals to the integral

$$\int_1^0 \sin^2(2\pi x)dx.$$

Explain why the numerical result is unacceptable. Suggest ways to improve the accuracy. [3]