Math 311-1 Test I
Fall 1999

Attempt at most five questions

1. (a) Perform two iterations of the secant method for the polynomial 
$p(x) = x^3 - 2x^2 + 1$ starting with $x_0 = -1, x_1 = 0$. [4 points]

(b) Comment on the convergence rate of the secant method. [2 points]

(c) Explain how quadratic convergence is related to the number of accurate digits. [2 points]

(d) Discuss in detail at least three stopping criteria that one may include when implementing the secant method. [6 points]

(e) Describe at least three situations where the secant method would break down. [6 points]

2. (a) Explain the following terms: mantissa, overflow, machine epsilon, non-associative addition and catastrophic cancellation. [10 points]

(b) Suppose that $f_l(y)$ is a $k$-digit chopping approximation to $y$. Find the least upper bound for

$$\left| \frac{y - f_l(y)}{y} \right|$$

3. (a) Consider the binary number

$$0.b_1 b_2 \ldots b_m$$

Devise an algorithm using Horner’s scheme to compute the decimal value of this number. [10 points]

(b) Let $f(x)$ be a function with a triple root at, say $x = a$,

i. Show that the iteration function corresponding to the Newton’s method is non-zero at the zero $x = a$. [3 points]

ii. Describe how the presence of multiple zeros effects the rate of convergence of Newton’s method. [2 points]

iii. Suggest a suitable modification to Newton’s method that will restore a quadratic rate of convergence near $x = a$. [2 points]

iv. Show that the iteration function corresponding to the modified Newton’s method is zero at the zero $x = a$. [3 points]
4. (a) Define the follow terms: [10 points]
  i. contraction;
  ii. rate of convergence of a fixed point iteration;
  iii. uniqueness condition for a fixed point iteration.
(b) Estimate how many steps of the bisection method one should apply to the equation

\[ x^3 = 2x^2 - 1 \]

over the interval \([1.4, 2]\) so that the absolute error is no bigger than 0.00001. [10 points]

5. For the equation

\[ x = 4^{-x} \]

(a) Determine an iteration function \( g(x) \) and an interval \([a, b]\) on which the fixed point iteration applied to \( g(x) \) will converge. [12 points]
(b) Estimate the number of iterations necessary to obtain approximations accurate to within \( 10^{-5} \) using an appropriate initial guess. [5 points]
(c) Explain the terms attractive and repulsive fixed points and their relation to the derivative value of a general iteration function \( g(x) \) at the fixed point. [3 points]

6. (a) Let \( a = 2/9 \). Describe the floating point number with three decimal digits obtained by chopping \( a \). Compute the absolute and relative errors. [4 points]
(b) Use Taylor series to derive a second order approximation to \( f'(a) \) based on the three values \( f(a - h), f(a) \) and \( f(a + 2h) \). Note that the value \( f(a + h) \) should not be used. [8 points]
(c) Compare the rate of convergence of the following limits:

\[ (a) \lim_{n \to \infty} \log(1 - \frac{1}{n^2}) = 0, \quad (b) \lim_{n \to \infty} n \left(1 - \cos\left(\frac{1}{n}\right)\right) = 0 \]

[8 points]