Splines

Spline: Continuity at knots

\[ S_0(t_0) = y_0 \]
\[ S_0(t_1) = S_1(t_1) = y_1 \]
\[ S_1(t_2) = S_2(t_2) = y_2 \]
\[ \vdots \]
\[ S_{n-2}(t_{n-1}) = S_{n-1}(t_{n-1}) = y_{n-1} \]
\[ S_{n-1}(t_n) = y_n \]

\[
\begin{array}{c|c}
1 + 2(n - 1) + 1 & 2n \\
\hline
n - 1 & n - 1 \\
\end{array}
\]

Continuity for derivatives:

\[ S_0'(t_1) = S_1'(t_1) \quad S_0''(t_1) = S_1''(t_1) \]
\[ S_0'(t_2) = S_1'(t_2) \quad S_0''(t_2) = S_1''(t_2) \]
\[ \vdots \]
\[ S_{n-2}'(t_{n-1}) = S_{n-1}'(t_{n-1}) \quad S_{n-2}''(t_{n-1}) = S_{n-1}''(t_{n-1}) \]

Total \( 4n - 2 \) conditions:
still have two degrees of freedom

Determination of spline functions

\[ S_i(x) = a_i + b_i x + c_i x^2 + d_i x^3, \quad x \in [t_i, t_{i+1}] \]

\[ S_i''(x) = 2c_i + 6d_i x \quad \text{(LINEAR)} \]

Let

\[
\begin{array}{c}
S_i'(t_i) = M_i \\
S_i'(t_{i+1}) = M_{i+1} \\
\end{array}
\]

\{ unknowns \}

then

\[ S_i''(x) = M_i \left( \frac{t_{i+1} - x}{t_{i+1} - t_i} \right) + M_{i+1} \left( \frac{x - t_i}{t_{i+1} - t_i} \right) \]

Integrate twice and fix constants of integration using \( S_i(t_i) = y_i, S_i(t_{i+1}) = y_{i+1} \):

\[ S_i(x) = \frac{M_i}{6} \frac{(t_{i+1} - x)^3}{t_{i+1} - t_i} + \frac{M_{i+1}}{6} \frac{(x - t_i)^3}{t_{i+1} - t_i} \]
Thus if \( M_i, M_{i+1} \) are determined, we have \( S_i(x) \).

**Determination of \( M_i \)'s**

Use continuity of derivative of spline function to find \( M_i \)'s

(continuity of second derivative already assumed)

\[
S'_{i-1}(t_i) = S'_i(t_i), \quad i = 1, 2, \ldots, n - 1
\]

Since

\[
S'_i(x) = \frac{M_i (t_{i+1} - x)^2}{2} \frac{t_{i+1} - t_i}{t_{i+1} - t_i} + \frac{M_{i+1} (x - t_i)^2}{2} \frac{t_{i+1} - t_i}{t_{i+1} - t_i}
+ \frac{6y_{i+1} - M_{i+1}(t_{i+1} - t_i)^2}{6(t_{i+1} - t_i)}
+ \frac{6y_i - M_i(t_{i+1} - t_i)^2}{6(t_{i+1} - t_i)}
\]

so

\[
S'_i(t_i) = \frac{t_{i+1} - t_i}{3} M_i - \frac{t_{i+1} - t_i}{6} M_{i+1} + \frac{y_{i+1} - y_i}{t_{i+1} - t_i}
\]

\[
S'_{i-1}(t_i) = \frac{t_i - t_{i-1}}{6} M_{i-1} + \frac{t_i - t_{i-1}}{3} M_i + \frac{y_i - y_{i-1}}{t_i - t_{i-1}}
\]

**Determination of \( M_i \)'s (cont'd)**

So for \( i = 1, 2, \ldots, n - 1 \):

\[
\frac{t_i - t_{i-1}}{6} M_{i-1} + \frac{t_{i+1} - t_i}{3} M_i + \frac{t_{i+1} - t_i}{6} M_{i+1}
= \frac{y_{i+1} - y_i}{t_{i+1} - t_i} - \frac{y_i - y_{i-1}}{t_i - t_{i-1}}
\]

Get \( n - 1 \) equations with \( n + 1 \) unknowns.

**Need end conditions:**

<table>
<thead>
<tr>
<th>natural</th>
<th>clamped</th>
<th>specified</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_0 = 0 )</td>
<td>( S'(t_0) = f'(t_0) )</td>
<td>( M_0 = f''(t_0) )</td>
</tr>
<tr>
<td>( M_n = 0 )</td>
<td>( S'(t_n) = f'(t_n) )</td>
<td>( M_n = f''(t_n) )</td>
</tr>
</tbody>
</table>

**Note.** Other end conditions possible:

- e.g. **not-a-knot** condition

Tridiagonal systems from spline functions
For natural splines get a symmetric, diagonally dominant tridiagonal system:

\[
A \begin{bmatrix}
M_1 \\
M_2 \\
\vdots \\
M_{n-1}
\end{bmatrix} = \begin{bmatrix}
v_1 \\
v_2 \\
\vdots \\
v_{n-1}
\end{bmatrix}
\]

where

\[
A = \begin{pmatrix}
u_1 & \frac{1}{6}h_1 & 0 & \cdots & 0 & 0 \\
\frac{1}{6}h_1 & u_2 & \frac{1}{6}h_2 & \cdots & 0 & 0 \\
0 & \frac{1}{6}h_{n-3} & u_{n-2} & \frac{1}{6}h_{n-2} & \cdots & 0 \\
0 & 0 & \frac{1}{6}h_{n-2} & u_{n-1}
\end{pmatrix}
\]

\[h_i = t_{i+1} - t_i, \quad u_i = \frac{1}{3}(t_{i+1} - t_{i-1})\]

\[b_i = (y_{i+1} - y_i)/h_i, \quad v_i = b_i - b_{i-1}\]

Natural Cubic Spline

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>3</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.67</td>
<td>0.5</td>
<td>0.33</td>
<td>0.2857</td>
<td></td>
</tr>
</tbody>
</table>

With \(M_0 = M_4 = 0\),

\[
\begin{pmatrix}
0.3333 & 0.083330 \\
0.08333 & 0.5 & 0.1667 \\
0.1667 & 1
\end{pmatrix}
\begin{bmatrix}
M_1 \\
M_2 \\
M_3
\end{bmatrix} =
\begin{bmatrix}
0.320 \\
0.170 \\
0.1479
\end{bmatrix}
\]

\(M_1 = 0.9238, \ M_2 = 0.1448, \ M_3 = 0.1237\)

In \([0,1.5]\),

\[S(x) = 1.429 + 0.1868 \times -0.9238 \times x^2 + 0.3079 \times x^3\]

In \([1.5,2]\),

\[t_i = 1.5, \ t_{i+1} = 2, \ y_i = 0.67, \ y_{i+1} = 0.5, \]

\(M_1 = 0.9238, \ M_{i+1} = 0.1448\)

\[S(x) = 3.345 - 3.645 \times x + 1.630 \times x^2 - 0.2597 \times x^3\]

In \([2,3]\),

\[S(x) = 1.295 - 0.5706 \times x + 0.09347 \times x^2 - 0.003513 \times x^3\]

In \([3,3.5]\),

\[S(x) = 1.479 - 0.7541 \times x + 0.1546 \times x^2 - 0.01031 \times x^3\]