Math 334 Ordinary Differential Equations

Lecture 1

Objectives
- Introduction to mathematical models described by differential equations

Sections covered
1.1

Vocabulary
Mathematical models, direction field, equilibrium solution

Concepts
- Mathematical models
- Math. model using differential equations
- Info from DE:
  - Direction field
  - equilibrium solution
- Constructing math models

Lecture 2

Objectives
- Solution of linear first order equations with constant coefficients
- Classification of differential equations

Sections covered
1.2, 1.3

Vocabulary
Initial condition, initial value problem, coefficients of differential equation, normalized coefficients, general solution, integral curves

Concepts
- solution of linear first order equations with constant coefficients
- asymptotic behavior of solutions
- order of an equation
- linear and nonlinear equations
- homogeneous and non-homogeneous equations
- system of first order equations

Lecture 3

Objectives
- Use integrating factor to solve first order linear equation with variable coefficients \( y' + p(t) y = g(t) \)

Sections covered
2.1

Vocabulary
Integrating factor

Concepts
- general form of first order linear equation with variable coefficients \( y' + p(t) y = g(t) \)
- construction of integrating factor
  \[ \mu(t) = \exp \left( \int p(t) \, dt \right) \]
- construction of general solution
  \[ y(t) = \frac{\int \mu(t) g(t) \, dt + C}{\mu(t)} \] (best done from start)
- construction of integral curves

Method properties:
Integrating factor \( \mu(t) = \exp \left( \int p(t) \, dt \right) \) may be hard to compute. Care needs to be taken when \( p(t) \) is negative.

Lecture 4

Objectives
- Identify and solve separable equations

Sections covered
2.2

Vocabulary
Separable equations, implicit form of solution
**Concepts**

- General first order equation \( y' = f(t, y) \) may be written in the form \( M(t, y) + N(t, y) y' = 0 \)
- Most general separable equation has form \( M(t) + N(y) y' = 0 \)
- Solution technique: find functions \( H(t), G(y) \) such that \( \frac{dH}{dt} = M \) and \( \frac{dG}{dy} = N \), then separable equation becomes \( H(t) + G(y) = C \), const
- First order linear equation with constant coefficient for \( y \): \( y' + p(t) y = g(t) \) is NOT separable

**Method properties:**

May get implicit form of solution instead of an explicit one

**Lecture 5**

**Objectives**

Discuss applications that can be modelled by first order equations, such as mixing problem, continuous compound interest etc

**Sections covered**

2.3

**Vocabulary**

**Concepts**

- conservation of mass in mixing problem
- continuous compound interest
- contaminant dispersal
- escape velocity

**Lecture 6**

**Objectives**

- Statement of existence and uniqueness of solutions of linear and nonlinear ODE-IVP
- Equation modelling exponential growth or decline or decay
- Logistic growth and qualitative description that depends on parameter values

**Sections covered**

2.4, 2.5

**Vocabulary**

**Concepts**

- existence and uniqueness theorem for linear ODE
- existence and uniqueness theorem for nonlinear ODE
- interval of definition: interval over which solution exists (may have non-uniqueness)
- exponential growth assumption: rate of change of population proportional to population size
- relation between half life and decay constant
- logistic equation assumption and derivation
- equilibrium solution and critical points
- qualitative behavior of solution without solving for it
- use of derivative information of slope function to provide better picture

**Lecture 7**

**Objectives**

More on equations related to logistic growth and qualitative description

**Sections covered**

2.5

**Vocabulary**

**Concepts**

- analytic solution of logistic equation
- stability
  - asymptotically stable solution
  - unstable equilibrium solution
- logistic equation with negative RHS
- qualitative behavior leading to concept of threshold level
- analytic solution of logistic equation with negative RHS
- logistic growth with a threshold level
Lecture 8

Objectives
Identify and solve exact equations

Sections covered
2.6

Vocabulary

Concepts
- Exact equation is an equation of the form $H'(t,y(t)) = 0$
- Equations of the form $M(t,y)+N(t,y) \, y' = 0$ is EXACT if one can find function $H(t,y)$ s.t. $dH/dt = M$ and $dH/dy = N$
- Exact equation has solution $H(t,y) = C$
- Theorem: equation is exact if and only if $dM/dy = dN/dt$ (assuming $M,N,dM/dy, dN/dt$ all continuous in some rectangle)
- If equation is not exact, it may become exact after multiplication by an integrating factor
- In general, integrating factor is difficult to compute.
- Special case: integrating factor depending on $x$ or on $y$ only
- Integrating factor is non-unique

Lecture 9

Objectives
- Discuss and outline proof of existence and uniqueness theorem of general first order equation

Sections covered
2.8

Vocabulary

Integral equation, Picard iteration/ method of successive approximation

Concepts
- Existence and uniqueness theorem: $f$ and $df/dy$ continuous in box
- integral equation
- Picard iteration/ method of successive approximation
- issues:
  - break down of iteration possible?
  - convergence of sequence of solutions
  - if convergent, is limit function a solution of ODE?
  - are there other solutions?

Lecture 10

Objectives
- Solution of second order linear homogeneous equations with constant coefficients using characteristic equations: ignoring repeated roots

Sections covered
3.1, 3.4, 4.2

Vocabulary

Characteristic equation

Concepts
- Using solution of the form $y = \exp(mt)$ leads to characteristic equation
- Solving for two roots
  - two real distinct roots
  - two complex conjugate roots
  - two repeated roots-- only one solution-- mention $\exp(m \, t)$ and $t \exp( m\, t)$
- General solution of form $C1 \, \exp( m1 \, t) + C2 \, \exp( m2 \, t)$
  - if $m1$ and $m2$ are complex, $C1$ and $C2$ must also be complex
  - $\exp( ix) = \cos x + i\, \sin x$ (from series of $\exp(x)$ )
- Alternate form for complex case: $y = A \, \cos w \, t + B \, \sin w \, t$
- Higher order linear homogeneous equations with constant coefficients -- same techniques

Lecture 11

Objectives
Introduce concepts of fundamental solutions of linear homogeneous equations

Sections covered
3.2, 4.1
**Vocabulary**
Differential operator notation, Wronskian, fundamental set of solutions

**Concepts**
- Differential operator notation \( L[\ ] \) for linear homogeneous equations
- Existence, uniqueness and smoothness theorem for second order linear (nonhomogeneous) equations
- The longest interval in which the solution of an IVP exists is the largest interval containing the initial point and lying between singular points of the normalized coefficients of the differential equation
- Principle of superposition
  - linear combination of solutions is also a solution
  - verification of principle of superposition leads to Wronskian
- Theorem: If the Wronskian of two solutions to a differential equation is non-zero, then the corresponding IVP has a solution.
- Theorem: all solutions are linear combination of solutions with non-zero Wronskian

**Lecture 12**

**Objectives**
- Explore connection between Wronskian and linear independence

**Sections covered**
3.3, 4.1

**Vocabulary**
Linearly dependent and independent solutions

**Concepts**
- Linear independent differentiable functions have non-zero Wronskian
- Show \( \exp(mt) \) and \( t \exp(mt) \) linearly independent
- Abel's theorem: computing Wronskian without knowing the solutions
  - For \( Ly= y''+py'+qy=0 \), Wronskian satisfies a first order equation \( W'+p(t)W= 0 \)
- Wronskian is either always zero or never zero
- Theorem: solutions to linear homogeneous equation are linear independent iff Wronskian is non-zero
- Fundamental set of solutions and linear independence for higher order equations

**Lecture 13**

**Objectives**
- Solve linear homogeneous equations for the case of repeated roots using D'Alembert's "variation of parameter" technique
- Reduction of order for linear homogeneous equations with non-constant coefficients

**Sections covered**
3.5, 4.2

**Vocabulary**

**Concepts**
- Solve linear homogeneous equations for the case of repeated roots using \( \exp(mt) \) and \( t \exp(mt) \)
- more generally, use D'Alembert's variation of parameter idea by setting \( y_2(t)= v(t) y_1(t) \)
- Reduction of order for linear homogeneous equations with non-constant coefficients
- generalization to higher order equations: \( \exp(mt), t \exp(mt) t^2 \exp(mt) \) etc
- re-examine complex case in example 4.2.4

**Lecture 14**

**Objectives**
Method of Undetermined Coefficients

**Sections covered**
3.6, 4.3

**Vocabulary**
Particular solution, Method of undetermined coefficients
Concepts

- **Theorem:** Difference of solutions of nonhomogeneous equation is a linear combination of fundamental solutions of homogeneous solutions.
- **Theorem:** General solution of nonhomogeneous equation is general solution of homogeneous equation plus particular solution.
- **Method of undetermined coefficients:** a method to find particular solution -- assume solution takes a certain form but with undetermined constant coefficients.

<table>
<thead>
<tr>
<th>Non-homogeneous term</th>
<th>Form of particular solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exponential function</td>
<td>Constant times same exponential function</td>
</tr>
<tr>
<td>Sine or cosine</td>
<td>Linear combination of sine and cosine</td>
</tr>
<tr>
<td>Polynomial</td>
<td>Polynomial of like degree</td>
</tr>
</tbody>
</table>

- Plug in the guess into the differential equation, and compare terms.
- The corresponding coefficients for each term will lead to an linear algebraic equation.
- Solve the system of linear equations for the coefficients.
- It may be necessary to multiply the above form by \( t^s \) where \( s=0,1 \) or 2 is the smallest integer such that all terms in the particular solution are not a homogeneous solution.
- If the non-homogeneous term \( g \) may be split into two terms \( g_1 + g_2 \), then the particular solution may be found by adding particular solutions \( y_1 + y_2 \) where \( L[y_1]=g_1 \) and \( L[y_2]=g_2 \).
- **General procedure for using the method of undetermined coefficients**:  
  - Find homogeneous solutions.
  - Split the non-homogeneous term \( g \) for form \( n \) subproblems \( L[y]=g_i \) if necessary.
  - For each subproblem, use the method of undetermined coefficient to find a particular solution \( y_i \).
  - Final particular solution is the sum of all the particular solutions \( y_i \).
- **General solution is sum of homogeneous solution and particular solution**.
- **If necessary, apply initial conditions to determine values of arbitrary constant in the homogeneous part of the general solution**.
- **Generalization to higher order equations OK**

**Method properties:**

Method is rather restrictive -- mainly for equations with constant coefficients in the homogeneous part and simple nonhomogeneous functions consisting of polynomials, exponential functions, sine and cosine functions. Other nonhomogeneous terms require other methods such as variation of parameters.

**Lecture 15**

**Objectives**

Variation of parameters

**Sections covered**

3.7, 4.4

**Vocabulary**

**Concepts**

- Starting with fundamental set of solutions \( (y_1, y_2) \) to the homogeneous equation, construct particular solution \( Y(t)=u_1(t)y_1+u_2(t)y_2 \). (variation of parameter)
- Assume \( u_1'y_1+u_2'y_2=0 \) to simplify \( Y'(t) \).
- Plug \( Y(t) \) into nonhomogeneous equation to get a second differential equation for \( u_1 \) and \( u_2 \), then solve for \( u_1 \) and \( u_2 \).
- **Theorem:** variation of parameter works if coefficients are continuous.
- **Extension to higher order equations possible.**
Lecture 16

Objectives
- To understand physically diverse phenomena may have very similar mathematical model
- To study damping in vibration

Sections covered
3.8

Vocabulary
Period, natural frequency of a vibration, amplitude, phase, quasifrequency, quasiperiod, critical damping, overdamping, underdamping,

Concepts
- undamped unforced vibration: simple harmonic motion
- solution may be written in polar form with physical interpretation
- damped unforced vibration gives quasi-periodic motion
- as damping parameter varies, qualitative feature of solution changes
- critical damping: damping parameter= 2 times the square root of product of spring constant and mass
- electric circuit has similar model

Lecture 17

Objectives
- To explore physical phenomena associated with forced vibration

Sections covered
3.9

Vocabulary
Beats, amplitude modulation,

Concepts
- Use of an external periodic force leads to generation of
  - beat if the external period does not match the natural period
  - resonance if the external period matches the natural period

Lecture 18

Objectives
Construct series solutions in equations near an ordinary point

Sections covered
5.1, 5.2

Vocabulary
Ordinary point, singular point, recurrence relation, Airy function

Concepts
- use of index in summation notation: shifting
- if a power series is zero, each coefficient is zero: this comparison leads to a recurrence relation
- use initial value and recurrence relation for find general form of coefficients

Method properties:
May have more than one general form for the coefficients: e.g. all even terms may have one form and all odd terms may have another form.

Lecture 19

Objectives
Construct series solutions in equations without singular point

Sections covered
5.3

Vocabulary
Ordinary point (more general definition), singular point,

Concepts
- Distinguish ordinary points and singular points
- Theorem:
  1. Linear second order equation with polynomial coefficients has a series
solution expansion near an ordinary point which may also be expressed as a linear combination of two linearly independent series solutions $y_1$ and $y_2$.

2. The radius of convergence of the two series solutions $y_1$ and $y_2$ are at least as large as the minimum of the radii of convergence of the series for the normalized coefficients.

- Use series expansion to find the two linearly independent series solutions
- Use the theorem to determine lower bound for the radius of convergence of series solution

**Lecture 20**

**Objectives**

Determine regular singular points and solution of Euler equations

**Sections covered**

5.4, 5.5

**Vocabulary**

Regular singular points, irregular singular points, Euler equations

**Concepts**

- Determination of singular points from coefficient polynomial of second derivative term
- Identification of regular singular points from product of power of $(x-x_0)$ and normalized coefficient polynomials: want find limits of $(x-x_0)Q(x)/P(x)$ and $(x-x_0)^2R(x)/P(x)$
- Identify form of Euler equation
- Using function of the form $y=x^r$ to get characteristic equation for Euler equations
  - Real distinct case straight forward
  - Repeated root case : set derivative (w.r.t. $r$) of $L[x^r]$ to zero to get second solution of form $y_2=x^r \log(x)$
  - Complex conjugate pair: write $x^r$ in terms of exponential and log functions, then apply Euler’s formula to get real fundamental solutions of the form $x^{\text{real}(r)}$ times sine or cosine of $(\text{imag}(r) \log(x))$
- Qualitative behavior of the solution to Euler equations

- Real distinct roots: behavior of positive and negative exponents
- Repeated root case: behavior of positive and negative exponents
- Complex root case: behavior with
  - Negative real part
  - Positive real part

**Lecture 21**

**Objectives**

Series solutions near a regular singular point

**Sections covered**

5.6, 5.8 (Order zero case)

**Vocabulary**

exponent at the singularity, indicial equation, Bessel equation, order of Bessel equation

**Concepts**

- Power series expansion of general second order equation sometimes lead to Euler equation
- For general second order equation, seek solution in the form of “Euler solution” times power series to get indicial equation
- Solve the indicial equation to get the exponents at the singularity
- Once the exponents are known, construct recurrence relation(s) for the power series coefficients
- Determine the radius of convergence of the power series if possible
- Identify form of Bessel equation
- Application of general power series solution technique to Bessel equation of order zero lead to two linearly independent solutions $J_0$ and $y_2$
- Bessel function $Y_0$
- Asymptotic behavior of Bessel functions for large $x$
- Asymptotic approximations of Bessel functions for large $x$