MATH 214

Chapter 16: Vector Calculus

0.1 Line Integrals

Consider a smooth plane curve \(C\) in the space given by the parametric equations
\[
x = x(t), \quad y = y(t), \quad z = z(t), \quad a \leq t \leq b
\]
(1)

or \(\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}\). Then, \(\mathbf{r}'\) is continuous and \(\mathbf{r}'(t) \neq 0\).

Construct a uniform partition of \([a, b]\) into \(n\) subintervals \([t_{i-1}, t_i]\) with a point \(t_i^*\) in it. Show Fig 1 book for corresponding partition along arc length parameter \(s\).

0.2 Definition 0.1: Line integral of Scalar Function \(f\) along \(C\)

- \(f\) is defined on smooth curve \(C\) given by equations (1)

Then,
\[
\int_C f(x, y, z) \, ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*) \Delta s_i
\]
is the line integral of \(f\) along \(C\), if this limit exists.

0.3 Theorem 0.1: Evaluation of Line Integral of a Scalar Function \(f\) along \(C\)

Discuss: Evaluation of arc length of curve \(C\) between \(a\) and \(b\).

The line integral of \(f\) along \(C\) can be evaluated as
\[
\int_C f(x, y, z) \, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2} \, dt
\]
0.4 Theorem 0.2: Line Integral of a Scalar Function $f$ along $C$
with respect to $x$, $y$, and $z$

\[
\int_C f(x, y, z)\,dx = \int_a^b f(x(t), y(t), z(t))\,x'(t)\,dt
\]

\[
\int_C f(x, y, z)\,dy = \int_a^b f(x(t), y(t), z(t))\,y'(t)\,dt
\]

\[
\int_C f(x, y, z)\,dz = \int_a^b f(x(t), y(t), z(t))\,z'(t)\,dt
\]

0.5 Definition 0.2: Vector Field

If $E \subseteq \mathbb{R}^3$, then a vector field on $\mathbb{R}^3$ is a function $F$ that assigns to each point $(x, y, z)$ in $E$ a three-dimensional vector $F(x, y, z)$. It can be expressed as

\[ F(x, y, z) = P(x, y, z)i + Q(x, y, z)j + R(x, y, z)k \]

0.6 Definition 0.3: Flow Lines or Streamlines

The flow lines of a vector field $F$ are the curves $C$ in the space (or plane) such that the vectors in the vector field are tangents to these curves.

Alternative Definition:
The flow lines or streamlines of a vector field are the paths followed by particles whose velocity field is the given vector field.

More precisely, if

\[ F(x, y, z) = P(x, y, z)i + Q(x, y, z)j + R(x, y, z)k \]

and a flow line curve $C$ has the parametric representation $r(t) = (x(t), y(t), z(t))$ then the components of $r$ satisfy the differential equation

\[ \frac{dx}{dt}(t) = P(x(t), y(t), z(t)) \quad \frac{dy}{dt}(t) = Q(x(t), y(t), z(t)) \quad \frac{dz}{dt}(t) = R(x(t), y(t), z(t)) \]

0.7 Definition 0.4: Work Done to Move Particle with Force $F$
along $C$

\[ W = \int_C F(x, y, z) \cdot T(x, y, z)\,ds = \int_C F \cdot T\,ds \]

where, $T(x, y, z)$ is the unit tangent vector at the point $(x, y, z)$ on $C$. 
0.8 Theorem 0.3: Work Done to Move Particle with Force $F$ along $C$, Using a parametric Representation of $C$

$$W = \int_{a}^{b} F(x(t), y(t), z(t)) \cdot r'(t) \, dt = \int_{C} F \cdot dr$$

*Show Why?*

0.9 Definition 0.5: Line Integral of a Vector Field $F$ along $C$

- $F$ is a continuous vector field defined on $C$
- $C$ is a smooth curve given by $r(t)$, $a \leq t \leq b$
Then, the line integral of the vector field $F$ along $C$ is given by

$$\int_{C} F \cdot dr = \int_{a}^{b} F(r(t)) \cdot r'(t) \, dt = \int_{C} F \cdot T \, ds$$

0.10 Alternative Representation of a Line Integral of a Vector Field $F$ along $C$

- If $F(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$
- $r(t) = \langle x(t), y(t), z(t) \rangle$
Then, $\int_{a}^{b} F(r(t)) \cdot r'(t) \, dt = \int_{C} P(x, y, z) \, dx + \int_{C} Q(x, y, z) \, dy + \int_{C} R(x, y, z) \, dz$

*Show Why?*

0.11 Theorem 1: Integration of Conservative Vector Fields

- $C$ is a smooth curve given by $r(t)$ where $a \leq t \leq b$,
  . $r(a) = \langle x_1, y_1, z_1 \rangle$, and $r(b) = \langle x_2, y_2, z_2 \rangle$
- $f$ is defined on a domain $D$ containing $C$.
- $f$ is differentiable and its gradient vector $\nabla f$ is continuous on $C$. Then,

$$\int_{C} \nabla f \cdot dr = f(r(b)) - f(r(a)) = f(x_2, y_2, z_2) - f(x_1, y_1, z_1)$$

*Work on proof.*
0.12 Alternative for Theorem 1: Integration of Conservative Vector Fields

- \( F \) is continuous on \( D \subseteq \mathbb{R}^3 \),
- \( D \) contains a smooth curve \( C \) given by \( r(t) \) where \( a \leq t \leq b \),
  \[ r(a) = \langle x_1, y_1, z_1 \rangle, \text{ and } r(b) = \langle x_2, y_2, z_2 \rangle \]
- \( F \) is a conservative vector field in the domain \( D \). It means there is \( f \) such that \( F = \nabla f \).

Then,
\[
\int_C F \cdot dr = \int_C \nabla f \cdot dr = f(r(b)) - f(r(a)) = f(x_2, y_2, z_2) - f(x_1, y_1, z_1)
\]

Work on proof and discuss examples

0.13 Definition 1: Independence of Path

- \( F \) continuous vector field on \( D \).
- \( C_1 \) and \( C_2 \) two curves or paths contained in \( D \).
- \( C_1 \) and \( C_2 \) have the same initial and terminal point.

Then, the line integral \( \int_C F \cdot dr \) is **independent of path** if

\[
\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr
\]

0.14 Theorem 2:

\( \int_C F \cdot dr \) is independent of path in \( D \) if and only if \( \int_C F \cdot dr = 0 \) for every piecewise-smooth closed path \( C \) in \( D \).

Discuss proof.

A closed path is one for which its terminal point coincides with its initial point.

0.15 Corollary:

If \( F \) is a conservative vector field defined on \( D \) then, the line integral \( \int_C F \cdot dr \) is independent of path in \( D \).

Is the reciprocal statement true?
0.16 Definition 2: Open and Connected Sets

- $D$ in $\mathbb{R}^{2,3}$ is open if for every point $P$ in $D$ there is a disk (or ball) with center on $P$ that is contained in $D$.
- $D$ is a connected set if any two points in $D$ can be joined by a curve contained in $D$.

0.17 Theorem 3

- $\mathbf{F}$ is a continuous vector field on an open connected region $D$.
- $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in $D$.

Then $\mathbf{F}$ is a conservative vector field on $D$ (there is $f$ such that $\nabla f = \mathbf{F}$)

0.18 Theorem 4: Property of a Conservative Vector Field.

- $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ is a conservative vector field on $D$.
- $P$ and $Q$ have continuous first-order partial derivatives on the domain $D$.

Then, for all $(x, y)$ in $D$

$P_y(x, y) = Q_x(x, y)$

work on proof

0.19 Theorem 5: Easy Way to Identify a Conservative Vector Field.

- $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ is a vector field defined on an open and simply connected region $D$.
- $P$ and $Q$ have continuous first-order partial derivatives on the domain $D$.

If $P_y(x, y) = Q_x(x, y)$ for all $(x, y)$ in $D$, then, $\mathbf{F}$ is conservative.

How can we find the potential $f$ corresponding to a conservative vector field?

Show examples
0.20 Theorem 6: Green’s Theorem

– $C$ is positively oriented, piecewise smooth, simple closed curve in the plane and is the boundary of a region $D$.
– $P$ and $Q$ have continuous first order partial derivatives on an open region that contains $D$. Then,
\[ \int_{C} P \, dx + Q \, dy = \int \int_{D} (Q_{x} - P_{y}) \, dA \]

0.21 Corollary 1: Green’s Theorem for Regions with a Hole

– If $D$ is a region enclosed by two simple and piecewise smooth curves $C_1$ and $C_2$
– $C_2$ is contained in the region enclosed by $C_1$.
– $C_1$ is positively oriented (counterclockwise for this case), and $C_2$ has the same orientation as $C_1$
– $P$ and $Q$ have continuous first order partial derivatives on an open region that contains $D$.
Then,
\[ \int \int_{D} (Q_{x} - P_{y}) \, dA = \int_{C_1} P \, dx + Q \, dy - \int_{C_2} P \, dx + Q \, dy \]

0.22 Corollary 2: Line Integral Over Complex Curves

– If $D$ is a region enclosed by two simple piecewise and smooth curves $C_1$ and $C_2$
– $C_2$ is contained in the region enclosed by $C_1$.
– $C_1$ is positively oriented (counterclockwise for this case), and $C_2$ has the same orientation as $C_1$
– $P$ and $Q$ have continuous first order partial derivatives on an open region that contains $D$.
– $P_{y}(x, y) = Q_{x}(x, y)$ on the open region that contains $D$. Then,
\[ \int_{C_1} P \, dx + Q \, dy = \int_{C_2} P \, dx + Q \, dy \]
0.23 Definition 3: Parametric Surfaces

Consider a region $D$ in the $uv$-plane and functions $x = x(u, v)$, $y = y(u, v)$, and $z = z(u, v)$ defined on $D$. Then, the vector-valued function defined on $D$ as

$$\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$$

is called a parametric equation of the parameters $u$ and $v$ and the set of points

$\{(x(u, v), y(u, v), z(u, v) : (u, v) \in D\}$ is called a parametric surface $S$ corresponding to the vector function $\mathbf{r}$.

0.24 Definition 4: Normal Vector to a Parametric Surface

Consider a parametric surface $S$ given by (2) defined on $D$. and $x = x(u, v)$, $y = y(u, v)$, and $z = z(u, v)$ have partial derivatives on $D$ and

$$\mathbf{r}_u \times \mathbf{r}_v = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \times \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle \neq 0,$$

on $D$

Then, the surface $S$ is smooth and the normal vector to its tangent plane at $(x_0, y_0, z_0) = \mathbf{r}(u_0, v_0)$ is defined as $\mathbf{r}_u \times \mathbf{r}_v(u_0, v_0)$

0.25 Definition 5: Surface Area of S

If $S$ is smooth parametric surface given by (2). Then, the surface area of $S$ is defined as

$$A(S) = \int \int_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA$$

0.26 Theorem 7: Surface Area of the Graph of $z = f(x, y)$

For $z = f(x, y)$, we can define the corresponding parametric equations as $x = x \quad y = y$, and $z = f(x, y)$.

If $f$ has continuous partial derivatives, then the surface area of $S$ is given by

$$A(S) = \int \int_D \sqrt{1 + z_x^2 + z_y^2} \, dA.$$
0.27 Definition 6: Surface Integral of a Scalar Function $f$

- If $f$ is continuous on a region $R$ containing the surface $S$.
- $S$ is a smooth surface and $r_u$ and $r_v$ are nonparallel in $D$.

Then, the surface integral of $f$ over the surface $S$ is defined as

$$\int \int_S f(x, y, z) \, dS = \int \int_D f(r(u, v)) \left| r_u \times r_v \right| \, dA \quad (5)$$

In particular, if $S$ is defined as $z = g(x, y)$, then

$$\int \int_S f(x, y, z) \, dS = \int \int_D f(x, y, g(x, y)) \sqrt{z_x^2 + z_y^2 + 1} \, dA \quad (6)$$

Work on Example 3 of the book.

0.28 Definition 7: Surface Orientation

A surface $S$ for which there is a continuously varying unit vector $n$ is called an oriented surface. The vector $n$ provides an orientation.

For a closed surface $S$ enclosing a solid $E$, positive orientation is defined as the one for which the normal vectors point out from $E$.

0.29 Definition 8: Surface Integral of Vector Fields

- $F$ is a continuous vector field defined on oriented surface $S$ with unit normal vector $n$,

then the surface integral of $F$ over $S$ is defined as

$$\int \int_S F \cdot S = \int \int_S F \cdot n \, dS \quad (7)$$

It is also called flux of $F$ across $S$.

If $S$ is given by the parametric function $r(u, v)$ then,

$$\int \int_S F \cdot S = \int \int_D F \cdot (r_u \times r_v) \, dA \quad (8)$$

Show this and work on Example 5 if time permits.
0.30 Theorem 8: Stoke’s Theorem

– S is an oriented piecewise-smooth surface, bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation.
– F is a vector field whose components have continuous partial derivatives on a region R that contains the surface S. Then,

\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_S \text{curl} \mathbf{F} \cdot d\mathbf{S} \]  \hspace{1cm} (9)

*Work on Example 1*

Green’s theorem is a special case of Stoke’s Theorem. In fact,

\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_S \text{curl} \mathbf{F} \cdot d\mathbf{S} = \int_S \text{curl} \mathbf{F} \cdot k \, dA \]  \hspace{1cm} (10)

0.31 Theorem 9: The Divergence or Gauss Theorem

– E is a simple solid region with surface boundary S.
– S has positive (outward) orientation.
– F is a vector field whose component functions have continuous partial derivatives on an open region R containing E. Then,

\[ \int \int \int_E \text{div} \mathbf{F} \, dV = \int \int_S \mathbf{F} \cdot d\mathbf{S} = \int \int_S \mathbf{F} \cdot \mathbf{n} \, dS, \]  \hspace{1cm} (11)

where \( \mathbf{n} \) is the unit outward normal vector to S

*Work on Example 2*