5.5 Self-Adjoint Operators and Sturm–Liouville EVP

Consider

$$I = \int_a^b \left[ u(x) \frac{d^2 u}{dx^2} + 4 u(x) \right] dx$$

Using integration by parts twice (assuming $u$ & $v$ self-smooth)

$$\int_a^b \left[ u(x) \frac{d^2 u}{dx^2} + 4 u(x) \right] dx = \int_a^b u(x) \frac{d^2 u}{dx^2} dx + 4 \int_a^b u(x) dx =$$

$$= \left[ u(x) \frac{d u}{dx} \right]^b_a - \int_a^b u(x) \frac{d^2 u}{dx^2} dx + 4 \int_a^b u(x) dx =$$

$$= uv \bigg|^b_a - u v' \bigg|^b_a + \int_a^b u v'' dx + 4 \int_a^b u dx =$$

$$= \left(vu' - uv'\right) \bigg|^b_a + \int_a^b u \left( v'' + 4v \right) dx$$

$$\Rightarrow \int_a^b \left[ u \left( v'' + 4v \right) - u \left( v'' + 4v \right) \right] dx = \left( vu' - uv' \right) \bigg|^b_a \tag{1.1}$$

Defining $L[u] \equiv \frac{d^2 u}{dx^2} + 4u$, where $L = \omega^2 + 4I$.

The last expression can be written as the differential operator.

$$\int_a^b \left( v L u \right)' - u L [v]' dx = \left[ vu' - uv' \right] \bigg|^b_a \tag{1.2}$$
Consider now

\[ I = \int_a^b v(x) \left[ u''(x) + u'(x) + u \right] \, dx \]

Using integration by parts (assuming \( v(x) \) and \( u(x) \) are sufficiently smooth)

\[ \int_a^b [u'' + u' + u] \, dx = \int_a^b vu'' \, dx + \int_a^b vu' \, dx + \int_a^b vu \, dx = \]

\[ = vu'' \bigg|_a^b - \int_a^b vu'' \, dx + vu' \bigg|_a^b - \int_a^b vu' \, dx + \int_a^b vu \, dx = \]

\[ = vu'' \bigg|_a^b - uv' \bigg|_a^b + \int_a^b vu'' \, dx + vu' \bigg|_a^b - \int_a^b vu' \, dx + \int_a^b vu \, dx = \]

\[ = \int_a^b \left[ u'' - u' + u \right] \, dx + \left. (vu'' - uv') \right|_a^b + vu' \bigg|_a^b \]

\[ \therefore \quad \int_a^b [v[u'' + u'] - u[v'' + v'] - v'' + v'] \, dx = \left[ vu'' - uv' \right] \bigg|_a^b + vu' \bigg|_a^b \]  

(2.1)

or defining the differential operator:

\[ \hat{\imath} [v] = v'' + v' + v \] and \( \hat{\imath} [u] = u'' + u' + u \).

we obtain

\[ \int_a^b \left[ u \hat{\imath} [v] - \hat{\imath} [v] \right] \, dx = \left[ vu'' - uv' \right] \bigg|_a^b + vu' \bigg|_a^b \]

(2.2)
There is an important difference between the two integral formulas (1.2) and (2.2).

That is in the differential operator obtained after integrating by part twice (in a way that all derivatives are removed from the function $u(x)$)

i) In (1.2) the new differential operator $\hat{L}$ acting over the function $V(x)$, is such that

$$\hat{L} = L.$$

ii) But in (2.1) this new differential operator $\hat{L}$ acting over $V(x)$ is

$$\hat{L} = \frac{d^2}{dx^2} - \frac{d}{dx} + I = \frac{d^2}{dx^2} + \frac{d}{dx} + I = L$$

The operator $L[u]=u''+4u$ is a particular case of the Sturm–Liouville operator

$$L[u] = [p(x)u'(x)]' + q(x)u(x).$$

Where $p(x) \equiv 1$, $q(x) \equiv 4$. 
However, the operator
\[ \hat{L}[u] = u'' - u' + u \]
is not of Sturm-Liouville type.

\[ \text{GREEN'S FORMULA.} \]

**Theorem.** If \( L[u] = (p(x)u'' + q(x)u) \) is the Sturm-Liouville operator and \( u(x) \in C^2[a,b] \) and \( p(x) \) is continuous then
\[ \int_a^b [L[u] - vL[u]] \, dx = p(x) \left[ uv' - vu' \right]_a^b \]

**Proof.**
\[ \int_a^b \left[ u \left( (pu)' + qu \right) - v \left( (pu)' + qu \right) \right] \, dx = \]
\[ = \int_a^b \left[ u (pu)' - v (pu)' \right] \, dx = \]
\[ = b u v' \left|_a^b \right. - \int_a^b p u' v' \, dx + \int_a^b p u' v' \, dx = \]
\[ = b u v' \left|_a^b \right. - \int_a^b p u' v' \, dx = \]
\[ \left. p(x) \left[ uv' - vu' \right] \right|_a^b \]

\[ \int_a^b [L[u] - vL[u]] \, dx = p(x) \left[ uv' - vu' \right]_a^b \]

This is called Green's formula.
The differential form of this formula is called Lagrange identity:

\[ u L [v] - v L [u] = \left[ \rho (x) \left( u'v - u v' \right) \right] \]  \hspace{1cm} (5.1)

**Definition.** If an operator \( L \) satisfies (4.1) for any two functions \( u(x), v(x) \in C^2 \) and \( u \) and \( v \) satisfy boundary conditions such that the right hand side of (4.1) is zero or

\[ \int_a^b \left[ u L [v] - v L [u] \right] dx = 0 \]  \hspace{1cm} (5.2)

then we will say that the operator \( L \) (corresponding to these boundary conditions) is **Self-Adjoint**.

**Remark 1.** If \( L \) is such that \( \hat{L} \) (obtained by interchanging \( u \) and \( v \)) is different than \( L \) and the BC's cancel the rhs, then \( \hat{L} \) is called the adjoint of \( L \) for these BC's.

**Remark 2.** A **self-adjoint operator** \( L \) is a concept similar to the concept of Symmetric matrix: \( A = A^T \)
**Theorem.** Consider the regular S-L EVP

\[
\begin{align*}
L \phi + \lambda \sigma(x) \phi &= [p(x) \phi']' + q(x) \phi + \lambda \sigma(x) \phi = 0 \quad (6.1) \\
\beta_1 \phi(a) + \beta_2 \phi'(a) &= 0 \quad (6.2) \\
\beta_3 \phi(b) + \beta_4 \phi'(b) &= 0 \quad (6.3)
\end{align*}
\]

The operator \( L \) defining (6.1) is self-adjoint for the boundary conditions (6.2) and (6.3): Dirichlet, Neumann, Robin or combinations of them.

**Proof.** In the Homework problems.

Application of the concept of self-adjoint operator

Orthogonality of eigenfunctions for the S-L EVP.

Let \( L \) = \( [p(x) u']' + q(x) u \). be the S-L operator and \( \phi_n \) and \( \phi_m \) two eigenfunctions corresponding to eigenvalues \( \lambda_n \) and \( \lambda_m \) with \( \lambda_n \neq \lambda_m \).

Then,

\[
\int_a^b \left( [\phi_n L \phi_n] - \phi_n L [\phi_m] \right) dx = \beta_1 \left( \phi_n \phi_n' - \phi_m \phi_m' \right) \bigg|_a^b = 0
\]

(Since \( \lambda_n \neq \lambda_m \))
Therefore,

\[ 0 = \int_{a}^{b} \left[ \phi_m (-\lambda_n \sigma \phi_n) - \phi_n (-\lambda_m \sigma \phi_m) \right] dx = \]

\[ = (\lambda_m - \lambda_n) \int_{a}^{b} \phi_n \phi_m \sigma(x) dx \]

Since \( \lambda_m \neq \lambda_n \) then,

\[ \int_{a}^{b} \phi_n(x) \phi_m(x) \sigma(x) dx = 0 \]

or \( \phi_n(x) \) is orthogonal to \( \phi_m(x) \) with weight \( \sigma(x) \).
Thus, all the eigenvalues for a regular SL-EVP are real.

**Proof.**

First, we will prove that if $\lambda$ is a complex eigenvalue with corresponding eigenfunction $\phi$, then $\lambda^*$ is also an eigenvalue with corresponding eigenfunction $\bar{\phi}$. In fact, if

$$L[\phi] + \lambda \sigma \phi = 0$$

$\Rightarrow$ the conjugate equation is

$$L[\bar{\phi}] + \bar{\lambda} \sigma \bar{\phi} = 0,$$

where $\sigma$ is real \hspace{1cm} (8.1)

Now, $L[\phi] = L[\bar{\phi}]$ (why?) $\Rightarrow$ $L(\phi) + \bar{\lambda} \sigma \bar{\phi} = 0$ \hspace{1cm} (8.1)

And $\bar{\phi}$ satisfies the same BC's that $\phi$ satisfies

$$\beta_1 \bar{\phi}(a) + \beta_2 \bar{\phi}'(a) = 0, \quad \beta_1, \beta_2 \text{ are real.}$$

And $\beta_3 \bar{\phi}(b) + \beta_4 \bar{\phi}'(b) = 0$

Therefore, $\bar{\phi}$ satisfies a SL-EVP with eigenvalue $\lambda^*$.

On the other hand, $\lambda + \lambda^*$ and the orthogonality condition is verified

$$(\lambda - \lambda^*) \int_a^b \phi \bar{\phi} \sigma dx = (\lambda - \lambda^*) \int_a^b |\phi|^2 \sigma dx = 0$$

$\Rightarrow \phi(x) \equiv 0$ Contradiction. Thus, $\lambda$ can't be complex.
Then- For each eigenvalue \( \lambda \), there is only one eigenfunction linearly independent.

Proof- Assume \( \phi_1(x) \) and \( \phi_2(x) \) are eigenfunctions corresponding to the same eigenvalue \( \lambda \) then,

\[
\phi_2 \ast \left[ L(\phi_1) + \lambda \sigma \phi_1 = 0 \right]
\]
\[
\phi_1 \ast \left[ L(\phi_2) + \lambda \sigma \phi_2 = 0 \right]
\]

\[
\phi_2 L(\phi_1) - \phi_1 L(\phi_2) = 0
\]

Using Lagrange identity

\[
0 = \phi_2 L(\phi_1) - \phi_1 L(\phi_2) = \frac{d}{dx} \left[ \rho \left( \phi_2 \phi'_1 - \phi_1 \phi'_2 \right) \right]
\]

\[
\Rightarrow \rho \left( \phi_2 \phi'_1 - \phi_1 \phi'_2 \right) \text{ constant}
\]

Clearly, if \( \begin{cases} \phi_2(a) = 0, & \phi_2(b) = 0 \\ \phi_1(a) = 0, & \phi_1(b) = 0 \end{cases} \text{ or } \begin{cases} \phi_2(a) = 0, & \phi_2(b) = 0 \\ \phi_1(a) = 0, & \phi_1(b) = 0 \end{cases} \)

then \( \left( \phi_2 \phi'_1 - \phi_1 \phi'_2 \right)(a) = 0 \)

\[
\Rightarrow \left( \phi_2 \phi'_1 - \phi_1 \phi'_2 \right)(b) = 0 \]

\[
\Rightarrow \frac{d}{dx} \left( \frac{\phi_2}{\phi_1} \right) = 0
\]

\[
\Rightarrow \phi_2 = C \phi_1 \Rightarrow \phi_2 \text{ and } \phi_1 \text{ linearly independent.}
\]

For Robin B.C. the theorem is also true.