7.3 Vibrating Rectangular Membrane

IBVP:

\[ u_{tt} = c^2 (u_{xx} + u_{yy}) = c^2 \nabla^2 u \]  \hspace{1cm} (1)

BC's

\[
\begin{align*}
    u(0,y,t) &= 0, & u(x,0,t) &= 0 & (2) \\
    u(L,y,t) &= 0, & u(x,H,t) &= 0 & (3)
\end{align*}
\]

IC's

\[
\begin{align*}
    u(x,y,0) &= \alpha(x,y) & (4) \\
    u_t(x,y,0) &= \beta(x,y) & (5)
\end{align*}
\]

Separation of variables:

\[ u(x,y,t) = h(t) \phi(x,y) \]

Substitution into (1) leads to

\[ h'' \phi = c^2 h \left( \phi_{xx} + \phi_{yy} \right) \]

Dividing by \( c^2 h \)

\[ \frac{h''}{c^2 h} = \frac{1}{c^2} \left( \phi_{xx} + \phi_{yy} \right) = -\lambda \]

Therefore, we obtain the two equations (ODE):

\[
\begin{align*}
    h'' - c^2 \lambda h &= 0 \\
    \phi_{xx} + \phi_{yy} &= -\lambda \phi
\end{align*}
\]

From the boundary conditions:

\[
\begin{align*}
    \phi(0,y) h(t) &= 0 \Rightarrow \phi(0,y) = 0 & \phi(0,0) h(0) = 0 \Rightarrow \phi(x,0) = 0 \\
    \phi(L,y) h(t) &= 0 \Rightarrow \phi(L,y) = 0 & \phi(x,H) h(t) = 0 \Rightarrow \phi(x,H) = 0
\end{align*}
\]
Two-dimensional eigenvalue problem:
\[
\begin{align*}
\nabla^2 \phi &= \phi_{xx} + \phi_{yy} = -\lambda \phi \quad (2.1) \\
\phi(0,y) &= 0, \quad \phi(x,0) = 0 \quad (2.2) \\
\phi(1,y) &= 0, \quad \phi(x,H) = 0 \quad (2.3)
\end{align*}
\]

The equation for \( h(t) \) has been solved for the 1-D problem
\[
h''(t) + \gamma h(t) = 0
\]
Assuming \( \lambda > 0 \) (oscillatory solutions),
\[
h(t) = C_1 \cos \gamma v \sqrt{t} + C_2 \sin \gamma v \sqrt{t} \quad (2.4)
\]
The eigenvalue problem is another PDE with homogeneous boundary conditions (2.2)–(2.3).

We try a further separation of variables in the variables \( x, y \)
\[
\phi(x,y) = f(x) g(y) \quad (2.5)
\]
Substitution of (25) into (2.1) leads to

\[ g(y) f''(x) + f(x) g''(y) = -\lambda f(x) g(y) \]

Dividing by \( f(x) g(y) \).

\[ \frac{f''(x)}{f} = -\lambda - \frac{g''(y)}{g(y)} = -\mu \]

Two ODE's result from this equation:

\[ f'' + \mu f = 0 \quad \text{and} \quad g'' + (\lambda - \mu) g = 0 \]

And using the BC's we obtain two eigenvalue problems:

\[ f'' + \mu f = 0 \quad \text{and} \quad f(0) = 0, \quad f(L) = 0 \quad (3.1) \]

And

\[ g'' + (\lambda - \mu) g = 0 \quad \text{and} \quad g(0) = 0, \quad g(\pi) = 0 \quad (3.2) \]

Eigenvalues for (3.1) are \( \mu_n = \left( \frac{n\pi}{L} \right)^2 \), \( n = 1, 2, \ldots \) (3.3)

And eigenfunctions are \( f_n(x) = \sin \left( \frac{n\pi}{L} x \right) \). (3.4)
For each \( n = 1, 2, \ldots \) (3.3) defines an eigenvalue problem:

\[
\begin{align*}
G''_n + (\lambda_{nm} - \mu_n) G_n &= 0 \\
G_n(c) &= 0, \quad G_n(H) &= 0
\end{align*}
\]

whose eigenvalues are

\[
\lambda_{nm} - \mu_n = \left(\frac{m\pi}{H}\right)^2, \quad m = 1, 2, \ldots
\]

and corresponding eigenfunctions are

\[
G_{nm}(y) = \sin\left(\frac{m\pi}{H} y\right), \quad m = 1, 2, \ldots
\]

Combining (3.3) and (4.1)

\[
\lambda_{nm} = \left(\frac{m\pi}{H}\right)^2 + \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots \quad m = 1, 2, \ldots
\]

and we have obtained the eigenvalues of the Eqs. (2.1)-(2.3).

From (2.5), (3.4) and (4.2), we also obtain the eigenfunctions:

\[
\phi_{nm} = \sin\left(\frac{n\pi}{L} x\right) \sin\left(\frac{m\pi}{H} y\right), \quad n = 1, 2, \ldots \quad m = 1, 2, \ldots
\]
Principle of Superposition:

Two families of product solutions combine to form the solution:

\[ U(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{nm} \sin \left( \frac{m \pi}{L} x \right) \sin \left( \frac{n \pi}{H} y \right) \cos \left( \sqrt{\frac{m^2 \pi^2 + n^2 \pi^2}{L^2 + H^2}} t \right) \]

\[ + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{nm} \sin \left( \frac{m \pi}{L} x \right) \sin \left( \frac{n \pi}{H} y \right) \sin \left( \sqrt{\frac{m^2 \pi^2 + n^2 \pi^2}{L^2 + H^2}} t \right) \]

(5.1)

The coefficients \( A_{nm} \) and \( B_{nm} \) can be determined from the IC's

\[ \alpha(x,0) = U(x,0,0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{nm} \sin \left( \frac{m \pi}{L} x \right) \sin \left( \frac{n \pi}{H} y \right) \]

Double Fourier Series.

There is a theory for the double F.S. similar to single F.S. However, it can be treated as two iterated single F.S.

We will assume that \( \alpha(x,y) \) and \( \beta(x,y) \) are at least piecewise smooth in each variable.
In fact,

\[ \alpha(x, y) = \sum_{m=1}^{\infty} \left( \sum_{n=1}^{\infty} A_{nm} \sin \left( \frac{n\pi x}{L} \right) \sin \left( \frac{m\pi y}{H} \right) \right) \]  
(6.1)

For \( x \) fixed,

\[ C_m(x) = \sum_{n=1}^{\infty} A_{nm} \sin \left( \frac{n\pi x}{L} \right) = \frac{2}{H} \int_0^H \alpha(x, y) \sin \left( \frac{m\pi y}{H} \right) dy \]

The equation (6.2) is the Fourier sine series of \( K(x) \)

\[ \text{then, } A_{nm} = \frac{2}{L} \int_0^L k(x) \sin \left( \frac{n\pi x}{L} \right) dx \]

or

\[ A_{nm} = \frac{2}{L} \left[ \frac{2}{H} \int_0^H \alpha(x, y) \sin \left( \frac{m\pi y}{H} \right) dy \right] \sin \frac{n\pi x}{L} dx \]

\[ = \frac{4}{LH} \int_0^L \int_0^H \alpha(x, y) \sin \left( \frac{m\pi y}{H} \right) \sin \left( \frac{n\pi x}{L} \right) \, dy \, dx. \]  
(6.3)

\( n = 1, 2, \ldots \)

\( m = 1, 2, \ldots \)
From \( U_t(x, y, 0) = \beta(x, y) \)

\[
U_t(x, y, 0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} \sqrt{\lambda_{mn}} \sin \left( \frac{m\pi x}{L} \right) \sin \left( \frac{n\pi y}{H} \right) = \beta(x, y)
\]

\[
C \sqrt{\lambda_{mn}} b_{mn} = \frac{4}{LH} \int_0^L \int_0^H \beta(x, y) \sin \left( \frac{m\pi y}{H} \right) \sin \left( \frac{n\pi x}{L} \right) \, dy \, dx
\]

(7.1)

Summarizing, the solution of (4) - (5) is given by (5.1) with the coefficients \( A_{mn} \) and \( B_{mn} \) defined by (6.3) and (7.1).

All this separation process

\( U(x, y, t) = \phi(x, y) h(t) \)

and later \( \phi(x, y) = f(x) g(y) \)

is equivalent to start from the beginning with

\( U(x, y, t) = f(x) g(y) h(t) \).
Modes of Vibrations.

Consider the IBVP:

\[
\begin{align*}
U_{tt} &= C^2 \nabla^2 U \\
U(0,y,t) &= 0, \quad U(x,0,t) = 0 \\
U(L,y,t) &= 0, \quad U(x,L,t) = 0 \\
U(x,y,0) &= \sin(\frac{\pi}{L}x) \sin(\frac{\pi}{H}y) \\
U_t(x,y,0) &= 0
\end{align*}
\]

\[
\text{Sinh:} \quad U(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[ A_{mn} \cos(c \sqrt{\lambda_{nm}} t) + B_{mn} \sin(c \sqrt{\lambda_{nm}} t) \right] \\
\times \sin(\frac{m\pi}{L}x) \sin(\frac{m\pi}{H}y)
\]

\[
\text{Using I.C.'s:} \quad U(x,y,0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin(\frac{m\pi}{L}x) \sin(\frac{m\pi}{H}y) = \sin(\frac{\pi}{L}x) \sin(\frac{\pi}{H}y)
\]

\[
\text{Using orthogonality:} \quad A_{11} = 1, \quad A_{mn} = 0, \quad \text{all other case.}
\]

\[
\text{From the 2nd I.C.} \quad U_t(x,y,0) = 0 \Rightarrow B_{mn} = 0, \quad \text{for all } n,m
\]

\[
U(x,y,t) = \cos(c \sqrt{\lambda_{11}} t) \sin(\frac{\pi}{L}x) \sin(\frac{\pi}{H}y)
\]
The previous Soln constitutes the most elementary mode of vibration. All other modes can be obtained

1) By defining initial conditions:
   \[ U(x,y,0) = \sin \left( \frac{m\pi}{L} x \right) \sin \left( \frac{n\pi}{A} y \right), \quad n=1,2,\ldots; \]
   \[ U_t (x,y,0) = 0 \]
   or
   
2) \[ U(x,y,0) = 0 \]
   \[ U_t (x,y,0) = \sin \left( \frac{m\pi}{L} x \right) \sin \left( \frac{n\pi}{A} y \right). \]