Numerical Solution of PDE

Homework #5. Winter 2005
Prof. Vianey Villamizar

1. Consider the IBVP for the heat conduction equation (similar to Problem 2 in Homework 3, but with non-homogeneous BC’s).

\[ \begin{cases} 
    u_t = \sigma u_{xx}, & 0 < x < 1, \ t > 0 \\
    u(x, 0) = \begin{cases} 
    2x, & 0 \leq x \leq \frac{1}{2} \\
    2(1 - x), & \frac{1}{2} \leq x \leq 1 
    \end{cases} \\
    u(0, t) = f(t), \quad u(1, t) = g(t) 
\end{cases} \quad (1) \]

If \( \sigma = 1 \), and \( f(t)=g(t)=0 \) the exact solution is given by

\[ u(x, t) = \sum_{k=1}^{\infty} \frac{8 \sin(k\pi/2)}{(k\pi)^2} e^{-k^2 \pi^2 t} \sin(k\pi x). \quad (2) \]

Write a computer code to solve the above IBVP for any Dirichlet BC’s. A good programming practice is to write codes using subroutines to perform the different calculations. I suggest that you construct your code according to: NumHeatWANhBC.m. You will find this in my web-page under project 5. The structure of NumHeatWANhBC.m is as follows: Grid Data, Grid in the xt-plane, Boundary Conditions, Initial Condition, Graph of Initial Temperature, Call to WeightedAverage subroutine (main algorithm), Call to a graphic subroutine, Call to Exact Solution Subroutine: EXSolnHeat (in case you have it), and Call to Error Subroutine: Error (in case you know the exact solution). You should have separate ”m-files” for IC and BC’s. I also posted in my web page a graphing subroutine: GraphingSubroutine that may be useful.

Inside your main solver WeightedAverage, you should call an independent tridiagonal subroutine to solve tridiagonal linear systems. You should construct this (as a separate file) as indicated in the textbook Section 4.1 pp. 6-8.

Perform the following experiments and show your results by Printing a graph of the surface in the xtu-space when the algorithm is converging. Otherwise, print your vector solution \( \mathbf{U}^N \) at certain level N to show that is diverging. Also, if the exact solution is known print the error in the \( l_2, \Delta x \)-norm and the \( ||.||_{inf} \)-norm. Construct a table showing the errors:

\[ l_{2, \Delta x} error = \sqrt{\sum_{j=1}^{N_x} |uex(j, Nt) - u(j, Nt)|^2} \Delta x \] and
\[ l_{inf} error = \max(\{|uex(:, Nt) - u(:, Nt)|\}) \]

1.1 Run one experiment using WA (Weighted-Average) with theta=0 (FT-CS explicit scheme) for the following data:

\( L = 1; \ tmax = 0.1; \ \Delta x = 0.05; \ r = 0.4; \ \sigma = 1; \ \Delta t = \Delta x^2 r / \sigma; \ Nt = \text{round}(tmax/\Delta t) + \)
1: \( N_x = \text{round}(L/\Delta x) + 1 \); \( \theta = 0; \text{jump} = 10; \)

Use homogeneous Dirichlet BC’s. You should observe stability. Then, run the same experiment with \( r=1 \) and \( t\text{max}=0.3 \). You should observe instability. Change your WA using \( \theta = 1 \) (BT-CS Euler implicit) keeping \( r=1 \). What do you observe? Explain. Use another \( \theta \), for example \( \theta = 1/2 \) (Crank-Nicholson). Comment on your result. From the results of all these experiments, construct a table showing the errors:

1.2 Use \( \theta = 1/3 \). Find \( r_0 \) such that for \( r \leq r_0 \) the WA is stable and unstable otherwise. Apply your WA with the same data as above, but \( r \leq r_0 \) and in another experiment with \( r > r_0 \). Comment on your results.

1.3 Run an experiment by changing your BC’s to \( f(t) = \sin(t) \) at the left end and \( g(t) = 3/4 \) at the right end. Use the following data:

\[
L = 1; \ t\text{max} = 7; \ \Delta x = 0.05; \ r = 10; \ \sigma = 1; \ \Delta t = \Delta x^2 \star r/\sigma; \\
N_t = \text{round}(t\text{max}/\Delta t) + 1; \ N_x = \text{round}(L/\Delta x) + 1; \ \theta = 1; \ \text{jump} = 5; \\
\]

Is your result consistent with the expected physical result? Explain.

1.4 Modify your code to include Neumann BC’s. Run a new experiment with \( u(0,t) = 0 \) and \( u_x(L,t) = 0 \). Use the following data:

\[
L = 1; \ t\text{max} = 0.5; \ \Delta x = 0.01; \ r = 10; \ \sigma = 1; \ \Delta t = \Delta x^2 \star r/\sigma; \ N_t = \text{round}(t\text{max}/\Delta t) + 1; \ N_x = \text{round}(L/\Delta x) + 1; \ \theta = 1; \ \text{jump} = 20; \\
\]

Make an physical interpretation of the surface graph obtained. Run again with the same data, but now increase \( t\text{max} = 1 \). Make comments.

2. Do problem (1) of Section 3.3 of the textbook. Include all its parts (1.1,1.2,1.3, and 1.4). Make comments on your results.