Summary: Conservation Laws

\( u(x,t) \): Density of physical variable. \( \{ \text{mass, energy, population} \} \)

PV: Physical Variable.

APV: Amount of physical variable.

\( \Phi(x,t) \): Flux of physical variable. \( D \): Closed Volume bounded by surface \( S \).

Definition: Flux of a physical variable is the amount of PV flowing through the domain bounding surface per unit of area and per unit of time.

Physical Units.

\( [u] = \frac{\text{APV}}{L^3} \), \( [\Phi] = \frac{\text{APV}}{L^2 \cdot t} \)
Conservation Law:

Domain $D$, bounded by surface $S$.

Rate of Change of the total APV inside $D$ = Net rate of flow across surface $S$ + Rate at which APV is introduced or taken out from domain $D$.

Mathematical representation

$$\frac{d}{dt} \int_D U(x,t) \, dv = - \int_S \hat{U}(x,t) \, \hat{n}(x) \, ds + \int_D f(x,t) \, dv$$  \hspace{1cm} (2.1)

Integral representation of the conservation law after applying the divergence theorem.

$$\int_D \left[ \frac{\partial U}{\partial t} (x,t) + \nabla \cdot \hat{U} - f(x,t) \right] \, dv = 0$$  \hspace{1cm} (2.2)

To transform (2.1) into (2.2), we need the functions $U$ and $\hat{U}$ to be sufficiently smooth. For example, we may require $U$ and $\hat{U}$ to be $C^1[\Omega \times \mathbb{R}]$. 
PDE obtained from integral representation.

\[
\frac{\partial u}{\partial t} (\mathbf{x},t) + \nabla \cdot \mathbf{\phi} (\mathbf{x},t) = f (\mathbf{x},t) \quad \mathbf{x} \in \Omega, \quad t > 0.
\]

I) Conservation of mass:

\( u(\mathbf{x},t) = \rho (\mathbf{x},t) \): density.

\( \mathbf{\phi} (\mathbf{x},t) \): mass flux \( \frac{\text{Mass}}{L^3 \cdot t} \)

Physically, mass is transported by particles of fluid flowing through the bounding surface \( S \) enclosing the volume \( D \). This process is called convection.

Notice that

\[
[\rho \mathbf{\bar{v}}] = \frac{M}{L^3 t} = \frac{M}{L^3 t}.
\]

Therefore, a good definition for the flux vector is given by

\( \mathbf{\phi} (\mathbf{x},t) \equiv \rho \mathbf{\bar{v}} \).
Substitution into (2.1) leads to

$$\frac{d}{dt} \int_\Omega p(x,t) \, dv = - \int_\Omega p(x,\hat{n}) \, ds + \int_\Omega f(x,t) \, dv \quad (4.1)$$

$$\int_\Omega p(x,\hat{n}) \, ds$$ is called Convective Surface term.

If there are not sources of mass inside $\Omega$ then $f(x,t) = 0$.

And (4.1) reduces to

$$\frac{d}{dt} \int_\Omega p(x,t) \, dv = - \int_\Omega p(x,\hat{n}) \, ds$$

Assuming $p(x,t)$ and $\vec{v}(x,t)$ are $C^1(\Omega)$ ($\text{PCE}$)

we obtain the PDE

$$\frac{\partial p}{\partial t} + \nabla \cdot (p \vec{v}) = 0$$

Continuity equation.

If flow is steady and incompressible

(4.1)
\[ p(x,t) \equiv \text{const.} \]

and eqn. (4.1) reduces to

\[ \nabla \cdot \vec{v} = 0 \]


Replacing \[ u(x,t) = \rho c T(x,t) \]

\( \rho \): density, \( c \): specific heat at constant Vol.

\( T \): temperature at \( x \) at time \( t \).

and \[ f(x,t) = -k_0 \nabla T \] Constitutive law

Known as Fourier's law of heat conduction.

into (3.1)

\[ \rho c \frac{\partial T}{\partial t} = -\nabla (-k_0 \nabla T) = k_0 \nabla^2 T \]

where \( f(x,t) \equiv 0 \), and \( \rho, c, \) and \( k_0 \) are constants properties.

or

\[ \frac{\partial T}{\partial t} = k \nabla^2 T \]

where \( k \equiv \frac{k_0}{\rho c} \) (5.1)

This is called Heat Conduction Equation.
If we consider a chemical mixture
\[ u(x, t) : \text{Density or concentration} \]
\[ \dot{u}(x, t) = -D \nabla u \]
Fick's law

Similar to Fourier's law

particles move from high concentration
to low concentrations.

Equation (3.1) for \( D \) constant and \( f \equiv 0 \) reduces to
\[ u_t = D \nabla^2 u \]

\( D \) is called diffusivity
\[ [D] = \frac{L^2}{t} \]
One dimensional heat conduction IBVP. \( \Omega = [0,1] \).

In order to solve (5.1), we need to impose an initial boundary condition (I.C.) and two boundary conditions (B.C.'s).

**I.C.** \( T(x,0) = f(x), \quad x \in (0,1) \).

Boundary conditions possibilities are

**Dirichlet Conditions:**

\[
T(0,t) = A(t) \\
T(1,t) = B(t)
\]

**Neumann Conditions:**

\[
\frac{\partial T}{\partial x} (0,t) = C(t) \\
\frac{\partial T}{\partial x} (1,t) = D(t)
\]

**Robin Conditions:**

\[
(\alpha_1 \frac{\partial T}{\partial x} + \beta_1 T)(0,t) = H(t) \\
(\alpha_2 \frac{\partial T}{\partial x} + \beta_2 T)(1,t) = G(t)
\]

or any combination of them.