1.1 Diffusion Equations.

1.1.1 Conservation laws.

They are balance laws

Thermodynamics: 1st Law

\[ \text{Change Internal Energy} = \text{Heat added} + \text{Work done on System}. \]

Population Biology:

\[ \text{Rate of Change of population} = \text{Birth rate} - \text{Death rate} + \text{Migration in and out of region}. \]

1-dim Conservation Law

\[ U(x,t): \text{density of physical variable} \]

\[ \text{Amount of Physical Variable: APV}. \]
Assumption: 1) \( u \) is constant in any cross section of tube

2) \( I = [a, b] \), arbitrary interval, of the real line \( R \).

\[
\text{Total amount of} \quad \Phi = \int_a^b u(x,t) \, dx
\]

P.V., \( x \in I \)

3) Motion of P.V. occurs in the axial direction

Define

\( \phi(x,t) \): Amount of the P.V. \( u \) flowing

\( \text{Flux through cross section at } x \text{ at time } t \), per unit of area, per unit of time.

\[
[\phi] = \text{A.P.V.} / \text{per unit of area} \times \text{per unit time}
\]

4) \( \phi > 0 \) if flow is in the positive \( x \)-direction.

\( \phi < 0 \) " " " in the negative \( x \)-direction.
Net rate of flow inside I:

Net rate of flow = \( A(a) \phi(a,t) - A(b) \phi(b,t) \) into I

Sources or sinks inside I: \( f(x,t,u) \).

\[ f(x,t,u) \] \( \text{Amount of P.V.} \)
\( \text{per unit vol} \times \text{per unit of time} \)

\( f > 0 \) is a source
\( f < 0 \) is a sink.

Conservation Law:

Rate of change of the total P.V. in I

\[ \frac{d}{dt} \int_{a}^{b} A(x) \ u(x,t) \ dx = A(a) \phi(a,t) - A(b) \phi(b,t) + \int_{a}^{b} f(x,t,u) A(x) \ dx \] (3.1)

Conservation Law in integral form.
Assuming that \( u, \phi, A \) are sufficiently smooth.

i) \( \int_a^b \frac{\partial}{\partial x} \left( A(x) \phi(x,t) \right) \, dx = A(b) \phi(b,t) - A(a) \phi(a,t) \).

ii) \( \frac{d}{dt} \int_a^b u(x,t) \, dx = \int_a^b \frac{\partial u}{\partial t} (x,t) \, dx \) (Assuming the cross sectional area does not change in time.

Subst. \( u \equiv \phi \).

\( \int_a^b \left[ \frac{\partial u}{\partial t} (x,t) + \frac{\partial}{\partial x} (A(x) \phi(x,t)) - A(x) f(x,t,u(x,t)) \right] \, dx = 0 \)

for any interval \( I = [a,b] \).

Then,

\( A(x) \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} (A \phi) = f \cdot A(x) \).

Assuming \( A(x) = \text{constant} \).

\( \frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial x} = f \), \( \forall x \in \mathbb{R}, \ t > 0 \).

Conservation law in integral form.
$u, \phi$ unknowns, but $f$ is given.

Higher Dimensions:

$V$: arbitary region of $\mathbb{R}^3$

$\partial V$: boundary (surface) of $V$.

Total amount of $P.V.$ in $V = \int_V \frac{U(\mathbf{x}, t)}{\nabla u} \, dV$

Flux is a vector $\mathbf{\phi}(\mathbf{x}, t)$

Net outward flux of $\mathbf{u}$ through $\partial V$ is given by

$$\int_{\partial V} \mathbf{\phi}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) \, ds.$$

Rate at which $u(\mathbf{x}, t)$ is produced or destroyed inside $V$

$$\int_V f(\mathbf{x}, t, u) \, dV$$
Conservation law in integral form:

\[ \frac{d}{dt} \int_V u \, dv = -\int_{\partial V} \phi \, \hat{n} \, ds + \int_V f \, dv \]  \hspace{1cm} (6.1) 

If \( u \) and \( \phi \) are smooth,

\[ \int_V \nabla \cdot \phi \, dv = \int_{\partial V} \phi \, \hat{n} \, ds \]

Substituting (6.1),

\[ \int_V \frac{\partial u}{\partial t} \, dv = -\int_{\partial V} \nabla \phi \, dv + \int_V f \, dv \]

Since \( V \) is arbitrary,

\[ \frac{\partial u}{\partial t} + \nabla \cdot \phi = f, \quad \forall \mathbf{x} \in \mathbb{R}^3, \quad t > 0. \]