Physical Interpretation of the Greens function.

Definition of delta function: \( \delta(x, x_0) \).

**Heat Conduction:** (1-D)

\[ U_t = kU_{xx} + Q(x), \quad 0 < x < L \]

If Steady: \( U_t = 0 \) and \( k = 1 \).

\[ \Rightarrow \quad -U'' = Q(x) \]

If BC's: \( U(0) = 0, \quad U(L) = 0 \).

\( Q(x) \): Represent a continuous distribution of sources along \( 0 < x < L \). At a given point \( x_0 \), the value \( Q(x_0) \) represent the strength of the source located at \( x_0 \).

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**Ideal Case:**

i) Source is located at one point \( x = x_0 \) only.

ii) This source has unit strength.

The following notation will be used:

\[ Q(x) = \delta(x, x_0) \]

i) \( \delta(x, x_0) = 0, \quad x \neq x_0 \) Not a conventional function, \( \delta(x) \).

ii) \( \int_0^L \delta(x, x_0) \, dx = 1 \) Any convolution goes everywhere except at one point \( x_0 \in (a, b) \), \( \int_a^b \delta(x, x_0) \, dx = 0 \).
Obtaining Green's function using the delta function: $\delta(x-x_0)$

Let's consider the steady state heat conduction operator again: $-u''(x)$.

And we will try to obtain the solution for a source of unit strength located at $x=x_0$.

\[
\begin{align*}
-\frac{d^2u}{dx^2} &= \delta(x-x_0) \\
\quad u(0) &= 0, \quad u(L) = 0.
\end{align*}
\]

We will obtain this directly.

The first thing to verify is that the associated S-L EVP does not have $\lambda = 0$ as an eigenvalue. In our case, we already know this. We expect that under this condition BVP (6.1) and (6.2) has a unique solution.

We start the procedure to obtain a solution. Solving two BVP governed by the corresponding homogeneous equation.

Choose $\chi_0$ s.t.

\[
\begin{cases}
\chi_0 \\
\chi_0^+
\end{cases}
\]

\[
\begin{align*}
a) \quad \begin{cases}
-\frac{d^2u}{dx^2} &= 0, \quad \chi_0 < x < x_0 \\
\quad u(0) &= 0, \quad u(x) \text{ specified later}
\end{cases} \\
\Rightarrow \begin{cases}
\quad \chi_0 = A_1 x_0 + B_1 \\
\quad u(x) = A_1 x + B_1
\end{cases}
\end{align*}
\]

\[
\begin{align*}
b) \quad \begin{cases}
-\frac{d^2u}{dx^2} &= 0, \quad x_0 < x < L \\
\quad u(L) &= 0, \quad u(x_0) \text{ specified later}
\end{cases} \\
\Rightarrow \begin{cases}
\quad \chi_0^+ = -\frac{B_2}{2} x + B_2 = B_2(1-\frac{x}{L}) \\
\quad u^+(x) = -\frac{B_2}{2} x + B_2
\end{cases}
\end{align*}
\]

\[
\begin{align*}
U(x) &= A_2 x + B_2 \quad 0 = u_0(0) = B_2 \\
U(x) &= A_2 x + B_2 \quad 0 = u_0(L) = A_2 L + B_2
\end{align*}
\]

\[
\begin{align*}
\Rightarrow A_2 &= -\frac{B_2}{L}
\end{align*}
\]

This can be shown directly by showing that the only soln. of $-u'' = 0, \quad u(0) = 0, \quad u(L) = 0$ is the trivial soln.
At $x = x_0$, we impose the condition that the temperature be continuous

$$U^+(x_0^+) = U^-(x_0^-)$$

This implies

$$A_1 x_0 = B_2 \left(1 - \frac{x_0}{L}\right) \Rightarrow \boxed{A_1 x_0 - (1 - \frac{x_0}{L}) B_2 = 0}$$

(7.1)

A second condition (Jump condition) is obtained by formally integrating the equation: between $x_0 - \epsilon$ and $x_0 + \epsilon$.

$$-\int_{x_0 - \epsilon}^{x_0 + \epsilon} u''(x) \, dx = \int_{x_0 - \epsilon}^{x_0 + \epsilon} \delta(x) \, dx = 1$$

$$\Rightarrow$$

$$\boxed{\left(U^+(x_0^+) - U^-(x_0^-)\right) = -1} \quad \text{Jump in the derivative.}$$

or

$$-\frac{B_2}{L} - A_1 = -1 \Rightarrow \boxed{A_1 = 1 - \frac{B_2}{L}}$$

Substitution into (7.1)

$$(1 - \frac{B_2}{L}) x_0 - (1 - \frac{x_0}{L}) B_2 = 0 \Rightarrow \boxed{B_2 x_0} \Rightarrow \boxed{A_1 = 1 - \frac{x_0}{L}}.$$

Therefore, the solution for the BVP (6.1)-(6.2) is given by

$$U(x, x_0) = \begin{cases} 
(1 - \frac{x_0}{L}) x, & x < x_0 \\
(1 - \frac{x}{L}) x_0, & x > x_0 
\end{cases}$$

(7.2)

$U(x, x_0)$ is symmetric. In fact, $U(x_0, x) = \begin{cases} 
(1 - \frac{x}{L}) x_0, & x \leq x_0 \\
(1 - \frac{x_0}{L}) x, & x > x_0 
\end{cases} = U(x, x_0)$.
Green's functions given by formula (7.23) are clearly symmetric. This symmetry is also called Maxwell reciopcity.

\[ G(x_1, x_2) = G(x_2, x_1) \]

The graphs of Green's functions (7.2) for sources located at \( x_1 = \frac{1}{5} \) and \( x_2 = \frac{1}{2} \) are as follows.

\[
G_1(x, \frac{1}{5}) = \begin{cases} 
(1 - \frac{1}{5})x, & x \leq \frac{1}{5} \\
(1 - x)\frac{1}{5}, & x > \frac{1}{5} 
\end{cases}
\]

\[
G_2(x, \frac{1}{2}) = \begin{cases} 
(1 - \frac{1}{2})x, & x \leq \frac{1}{2} \\
(1 - x)\frac{1}{2}, & x > \frac{1}{2} 
\end{cases}
\]

Therefore,

\[
G_1(\frac{1}{5}, \frac{1}{5}) = G_2(\frac{1}{5}, \frac{1}{2})
\]
Consider two functions \( u(x) \) and \( v(x) \) continuously differentiable twice.

\[
(u v')' = u' v' + u v''.
\]

\[
(v u')' = v' u' + v u''
\]

\[
(u v')'' - (v u')' = u v'' - v u''
\]

or \( u v'' - v u'' = (u v')' - (v u')' \)

Integrating between 0 and \( L \)

\[
\int_0^L (u v'' - v u'') \, dx = \int_0^L ((u v')' - (v u')') \, dx
\]

or

\[
\int_0^L (u v'' - v u'') \, dx = \left[ (u v')' - (v u')' \right]_0^L
\]

\[
= u(L) v'(L) - v(L) u'(L) - u(0) v(0) + v(0) u(0)
\]

So

\[
\int_0^L (u v'' - v u'') \, dx = u(L) v'(L) - v(L) u'(L) - u(0) v(0) + v(0) u(0)
\]
BVP with homogeneous boundary cond. and a forcing fn.

\[ -u''(x) = f(x), \quad u(0) = 0, \quad u(L) = 0. \]

Greens fn: \[ -G''(x,x_0) = \delta(x-x_0) \quad G(0,x_0) = 0, \quad G(L,x_0) = 0 \]

Green's formula:

\[ \int_0^L (uv'' - vu'') \, dx = [uv' - vu]_0^L \quad (11.1) \]

Replacing \( v = G \):

\[ \int_0^L u(x) G''(x,x_0) \, dx - \int_0^L G(x,x_0) u''(x) \, dx = \]

\[ = [u(x) G'(x,x_0) - G(x,x_0) u'(x)]_0^L \]

or

\[ \int_0^L u(x) (-\delta(x,x_0)) \, dx - \int_0^L G(x,x_0) (-f(x)) \, dx = 0 \]

\[ \Rightarrow \quad -u(x_0) = -\int_0^L G(x,x_0) f(x) \, dx \]

or

\[ u(x_0) = \int_0^L G(x,x_0) f(x) \, dx = \int_0^L G(x_0,x_0) f(x) \, dx \]

Interchanging roles \( x \leftrightarrow x_0 \)

\[ u(x) = \int_0^L G(x,x_0) f(x_0) \, dx_0 \quad (11.2) \]

\[ \rightarrow \quad \text{Go to physical interpretation on page } \# \text{ 8} \]
This is exactly the Green's function obtained previously by variation of parameters.

Now, we can describe the Green's function as the response of the system at a point \( x \), due to a source of unit strength located at \( x_0 \).

Comparison of the two solutions for the two BVPs:

1. \(-u'' = f(x), \quad 0 < x < L, \quad u(0) = 0, \quad u(L) = 0\)
2. \(-v'' = \delta(x - x_0), \quad 0 < x < L, \quad v(0) = 0, \quad v(L) = 0\)

For 1, we obtained

\[
\begin{align*}
\int_0^L G(x, x_0) f(x_0) \, dx_0 &= u(x).
\end{align*}
\]

(8.1)

For 2, we obtained

\[
\begin{align*}
v(x) &= \int_0^L G(x, x_0) \delta(x - x_0) \, dx_0 \\
&= G(x; x_0),
\end{align*}
\]

Interpretation:

Soln. of (1) corresponds to the response of the system to a concentrated source of unit strength located at \( x = x_0 \).

Soln. of (2) corresponds to the response of the system to a continuous distribution of sources on the interval \([0, L]\), each one of strength \( f(x_0) \) at \( x = x_0 \).
Boundary Value Problem with nonhomogeneous B.C.'s.

\[-u''(x) = f(x), \quad u(0) = \alpha, \quad u(L) = \beta.\]

\[-g''(x, x_0) = s(x, x_0), \quad g(0, x_0) = 0, \quad g(L, x_0) = 0.\]

Subst. into (11.1) Green's formula:

\[
\int_0^L u(x) (-s(x, x_0)) \, dx - \int_0^L g(x, x_0) (-f(x)) \, dx = \\
= \beta \frac{dg}{dx}(L, x_0) - \alpha \frac{dg}{dx}(0, x_0) = \\
= \beta \frac{dg}{dx}(L, x_0) - \alpha \frac{dg}{dx}(0, x_0)
\]

\[\Rightarrow \]

\[-u(x_0) + \int_0^L g(x, x_0) f(x) \, dx = \beta \frac{dg}{dx}(L, x_0) - \alpha \frac{dg}{dx}(0, x_0)\]

Using the symmetry of \(g(x, x_0): \quad g(x, x_0) = g(x_0, x)\) (12.1)

\[\Rightarrow \frac{dg}{dx}(x, x_0) = \frac{dg}{dx}(x_0, x). \quad (12.2)\]

Using (12.1) and (12.2) and reversing roles of \(x_0 \leftrightarrow x\) in (12.0),

We obtain

\[
u(x) = \int_0^L g(x, x_0) f(x_0) \, dx_0 - \beta \frac{dg}{dx_0}(x, L) + \alpha \frac{dg}{dx_0}(x, 0)
\]

\[\Rightarrow \]

\[
u(x) = \int_0^L g(x, x_0) f(x_0) \, dx_0 + \beta \frac{x}{L} + \alpha \left(1 - \frac{x}{L}\right) \quad (12.2)
\]