Generalized Green's Functions.

Consider the nonhomogeneous problem

\[
\begin{cases}
Au = f(x), & a < x < b \\
B_1 u(a) = 0, & B_2 u(b) = 0
\end{cases}
\]  

(1)  

(2)

when \( \lambda = 0 \) is an eigenvalue.

We want to find out the solution in terms of a "generalized Green's function".

We have seen that (1) and (2) has infinitely many solutions. Let's assume that \( \phi_h(x) \) is an eigenfunction corresponding to \( \lambda = 0 \), or equivalently that \( \phi_h(x) \) is a nontrivial solution of the corresponding homogeneous problem.

To have infinitely many solutions

\[
\int_a^b f(x) \phi_h(x) \, dx = 0
\]

\[
u = \sum_{n=1}^{\infty} u_n \phi_n
\]

Recall that

\[
u_n \lambda_n = f_n
\]

If \( \lambda_n = 0 \) \( \Rightarrow \) \( f_n = 0 \) for solutions to exist.
We will illustrate the process of construction of a Generalized Green's function and then of solutions of (1) and (2) through the following example:

\[
\begin{align*}
\frac{d^2 u}{dx^2} &= f(x) = x - \frac{L}{2}, \quad 0 \leq x \leq L, \\
\frac{du}{dx}(0) &= 0, \quad \frac{du}{dx}(L) = 0
\end{align*}
\]

Obviously, there are nontrivial solutions for the corresponding homogeneous problem: \( \phi_h(x) \equiv 1 \) and any multiple. Therefore, for a solution to exist,

\[
\int_0^L f(x) \phi_h(x) \, dx = \int_0^L f(x) \, dx = 0
\]

If \( f(x) = x - \frac{L}{2} \),

\[
\int_0^L \left( x - \frac{L}{2} \right) \, dx = \left[ \frac{x^2}{2} - \frac{Lx}{2} \right]_0^L = 0
\]

We observe that the Green's fn. for the BVP given does not exist. In fact, for \((x_0) \quad \frac{d^2 G}{dx^2} = \delta(x-x_0), \quad \frac{dG}{dx}(0) = 0, \quad \frac{dG}{dx}(L) = 0\) \(\star\)

\[
\int_0^L \delta(x-x_0) \, \phi_h(x) \, dx = \phi_h(x_0) \equiv 1 \neq 0
\]

So, \( \delta(x-x_0) \nmid \phi_h(x) \equiv 1 \). \(\Rightarrow\) There is no soln. (Green's fn.) for \((\star)\).
However, we can redefine rhs of (*) to achieve orthogonality as follows:

\[
\frac{d^2 G_m}{dx^2} = S(x - x_0) + C \phi_h(x)
\]

\(C\) is determined from the orthogonality condition

\[
0 = \int_0^L \phi_h(x) \left[ S(x - x_0) + C \phi_h(x) \right] dx = \int_0^L \phi_h(x) S(x - x_0) dx + C \int_0^L \phi_h(x)^2 dx
\]

\[
\Rightarrow 0 = \phi_h(x_0) + C \int_0^L \phi_h(x)^2 dx
\]

\[
\Rightarrow C = -\frac{\phi_h(x_0)}{\int_0^L \phi_h(x)^2 dx} \quad \text{because } \phi_h(x) \neq 0 \text{ nontrivial soln.}
\]

And the related B.V.P. (nonhomog.)

\[
\begin{cases}
\frac{d^2 G_m}{dx^2} = S(x - x_0) - \left( \phi_h(x_0) / \int_0^L \phi_h(x)^2 dx \right) \phi_h(x) \\
G_m(0) = 0, \quad G_m''(L) = 0
\end{cases}
\]

does not have nontrivial solutions, because rhs \(\bot \phi_h(x)\).
In our case, \( \phi_n(x) = 1 \)

\[ C = \frac{-1}{L} \]

And related BVP:

\[ \begin{cases} \frac{d^2 G_m}{dx^2} = S(x-x_0) - \frac{1}{L} \\ G_m(0) = 0, \quad G_m'(L) = 0 \end{cases} \]

If \( x \neq x_0 \),

\[ \frac{d^2 G_m}{dx^2} = -\frac{1}{L} \]

by integrating,

\[ \frac{dG_m(x)}{dx} = -\frac{1}{L}x + d_1x \]

To satisfy BC at the left,

\[ 0 = G_m'(0) = d_1 \Rightarrow \boxed{\ d_1 = 0 \} \]

To satisfy BC at the right,

\[ 0 = G_m'(L) = -\frac{L}{L} + d_4 \Rightarrow \boxed{\ d_4 = 1 \} \]

\[ G_m(x) = \begin{cases} -\frac{x}{L}, & x < x_0 \\ -\frac{x}{L} + 1, & x > x_0 \end{cases} \] \( \text{(**) \) }

The jump condition in the derivative at \( x = x_0 \)

\[ \frac{dG_m}{dx}(x_0^+) = \frac{dG_m}{dx}(x_0^-) = -\frac{1}{L} \left[ x_0 + \varepsilon - x_0 - \varepsilon \right] + 1 \]

\[ \varepsilon \to 0 \]

\[ G_m(x_0^+) - G_m(x_0^-) = 1 \]

It's automatically satisfied.
Integrating again (* *) \( \int dx \)

\[
G_m(x, x_0) = \begin{cases} 
-\frac{x^2}{2} + \frac{1}{L} + h_1(x_0) & x < x_0 \\
-\frac{x^2}{2} + \frac{1}{L} + x + h_2(x_0) & x > x_0 
\end{cases}
\]

Continuity at \( x_0 \) \( \Rightarrow \) \( G_m(x_0^+, x_0) = G_m(x_0^-, x_0) \).

\[
\Rightarrow -\frac{x_0^2}{2} + \frac{1}{L} + x_0 + h_2(x_0) = -\frac{x_0^2}{2} + \frac{1}{L} + h_1(x_0)
\]

\[
\Rightarrow h_1(x_0) - h_2(x_0) = x_0 \quad \Rightarrow h_1(x_0) = x_0 + h_2(x_0)
\]

or

\[
G_m(x, x_0) = \begin{cases} 
-\frac{x^2}{2} + \frac{1}{L} + x_0 + h_2(x_0) & x < x_0 \\
-\frac{x^2}{2} + \frac{1}{L} + x + h_2(x_0) & x > x_0 
\end{cases}
\]

\( h_2 \Rightarrow h_2(x_0) \)

by requiring symmetry: \( G_m(x, x_0) = G_m(x_0, x) \) \( \Rightarrow \)

\[
h_2(x_0) = -\frac{1}{L} \frac{x_0^2}{2} + \beta
\]

\[
G_m(x, x_0) = \begin{cases} 
-\frac{1}{L} \frac{x^2 + x_0^2}{2} + x_0 + \beta, & x < x_0 \\
-\frac{1}{L} \frac{x^2 + x_0^2}{2} + x + \beta, & x > x_0 
\end{cases}
\]
Now, a solution (among infinitely many) for the BVP given is
\[ U(x) = \int_{a}^{b} f(x_0) G_m(x, x_0) \, dx_0 \]

In fact, using Green's formula:
\[
\int_{a}^{b} [uLv - vLu] \, dx = \left[ p \left( \frac{u \, dv}{dx} - \frac{v \, du}{dx} \right) \right]_{a}^{b} + \int_{a}^{b} \left( \phi_h(x_0) \phi_h'(x_0) \right) \, dx_0 - G_m(x, x_0) f(x) \, dx = 0.
\]

Hence BC's
\[
L \left[ \phi_{h}(x_0) \right]
\]

\[ U(x_0) = \frac{\phi_{h}(x_0)}{\int_{a}^{b} \phi_{h}(x) \, dx} \int_{a}^{b} \phi_{h}(x) u(x) \, dx + \int_{a}^{b} G_m(x, x_0) f(x) \, dx.\]

Renaming variable, \( x_0 \leftrightarrow x \), and using symmetry of \( G_m(x, x_0) \)

\[ U(x) = \frac{\phi_{h}(x)}{\int_{a}^{b} \phi_{h}^2(x) \, dx} \int_{a}^{b} \phi_{h}(x_0) u(x) \, dx_0 + \int_{a}^{b} G_m(x, x_0) f(x) \, dx_0.\]

\[ \text{multiple of homog. soln.} \]

So particular soln.
\[ U(x) = \int_{a}^{b} G_m(x, x_0) f(x_0) \, dx_0 \]